Published by Institute of Physics Publishing for SISSA/ISAS
Received: *December 11, 2003*
Accepted: *February 4, 2004*

Matrix model as a mirror of Chern-Simons theory

Mina Aganagic,[a] Albrecht Klemm,[b] Marcos Mariño[a] and Cumrun Vafa[a]

[a] *Jefferson Physical Laboratory, Harvard University
Cambridge, MA 02138, U.S.A.*
[b] *Humboldt-Universität zu Berlin, Institut für Physik
D-10115 Berlin, Germany*
E-mail: mina@cartan.harvard.edu, aklemm@physik.hu-berlin.de, marcos@mail.cern.ch, vafa@physics.harvard.edu

Abstract: Using mirror symmetry, we show that Chern-Simons theory on certain manifolds such as lens spaces reduces to a novel class of hermitean matrix models, where the measure is that of unitary matrix models. We show that this agrees with the more conventional canonical quantization of Chern-Simons theory. Moreover, large-N dualities in this context lead to computation of all genus A-model topological amplitudes on toric Calabi-Yau manifolds in terms of matrix integrals. In the context of type-IIA superstring compactifications on these Calabi-Yau manifolds with wrapped D6 branes (which are dual to M-theory on G_2 manifolds) this leads to engineering and solving F-terms for $\mathcal{N} = 1$ supersymmetric gauge theories with superpotentials involving certain multi-trace operators.

Keywords: Chern-Simons Theories, String Duality, Matrix Models.

JHEP02(2004)010

http://jhep.sissa.it/archive/papers/jhep022004010/jhep022004010.pdf

Contents

1. Introduction

Recently it was observed in [1] that partition functions of Chern-Simons theory on certain manifolds can be represented as hermitean matrix integrals with a measure suitable for unitary matrix models. On the other hand, it was found in [2] that topological strings for B-branes are equivalent to hermitean matrix models. It is thus natural to ask if these two ideas are related. Since Chern-Simons theory arises from topological strings for A-branes [3] one is led to believe that the observation in [1] should be obtained by applying mirror symmetry to obtain certain B-brane matrix models. In this paper we will verify that this is indeed the case. For example by applying mirror symmetry to the deformed conifold $T^*\mathbf{S}^3$ we show that the Chern-Simons theory on $\mathbf{S}^3$ reduces to a gaussian hermitean matrix model with a unitary measure.

On the other hand the large-N transition proposed in [4], and derived from the worldsheet viewpoint in [5], relates Chern-Simons gauge theory to A-model topological strings (with or without branes) on certain non-compact Calabi-Yau threefolds. Thus the result we obtain here shows that the topological A-model on certain non-compact Calabi-Yau manifolds reduces to matrix integrals. In particular we consider the $\mathbb{Z}_p$ orbifold of the duality in [4] which suggests that Chern-Simons theory on lens space should be related to the $\mathbb{Z}_p$ quotient of the resolved conifold.[1] We find that the large N duality continues to hold upon orbifolding, and the choice of flat connection in the Chern-Simons theory on lens space maps to the extra blowup moduli from the twisted sectors on the closed string side.

This is a natural extension of the result that matrix integrals can compute intersection theory on moduli space of Riemann surfaces [7]. Moreover this sheds a new light on recent results [8, 9] which relate all genus open and closed topological A-model amplitudes with Chern-Simons theory. Namely, we can restate (and rederive) this result in terms of the equivalence of topological A-model and a suitable matrix model.

The matrix model we end up with is a novel kind of matrix model, in which the action is that of a hermitean matrix model $V(u)$, but the measure is that suitable for a unitary matrix $U = e^u$. This is not a unitary matrix model. In particular the action does not have the periodicity expected for a unitary matrix model. We explain how this arises from mirror symmetry. Moreover we are able to rewrite this in terms of an ordinary hermitean matrix model with the usual measure, at the expense of introducing multi-trace operators in the action.

From the viewpoint of type-IIA compactifications the A-branes which fill spacetime give rise to $\mathcal{N} = 1$ supersymmetric gauge theories. For example N D6 branes wrapped on $\mathbf{S}^3 \subset T^*\mathbf{S}^3$ gives rise, in the infrared, to pure $\mathrm{U}(N)$ Yang-Mills theory. However the F-terms of the full theory differ from that of pure Yang-Mills. Here we find, using this rewriting of the measure, that the theory can be viewed as a deformed $\mathcal{N} = 2$ theory with a mass term for the adjoint $m \operatorname{Tr} \Phi^2$, together with certain multi-trace operators of the form $S \operatorname{Tr} \Phi^k \operatorname{Tr} \Phi^l$ where S is the glueball field $S = \operatorname{Tr} \mathcal{W}^2$. Thus we can capture the deviations

[1] This idea has been advanced by a number of physicists, including R. Gopakumar, S. Sinha, E. Diaconescu, A. Grassi, B. Pioline, J. Gomis and E. Cheung. See also [6].

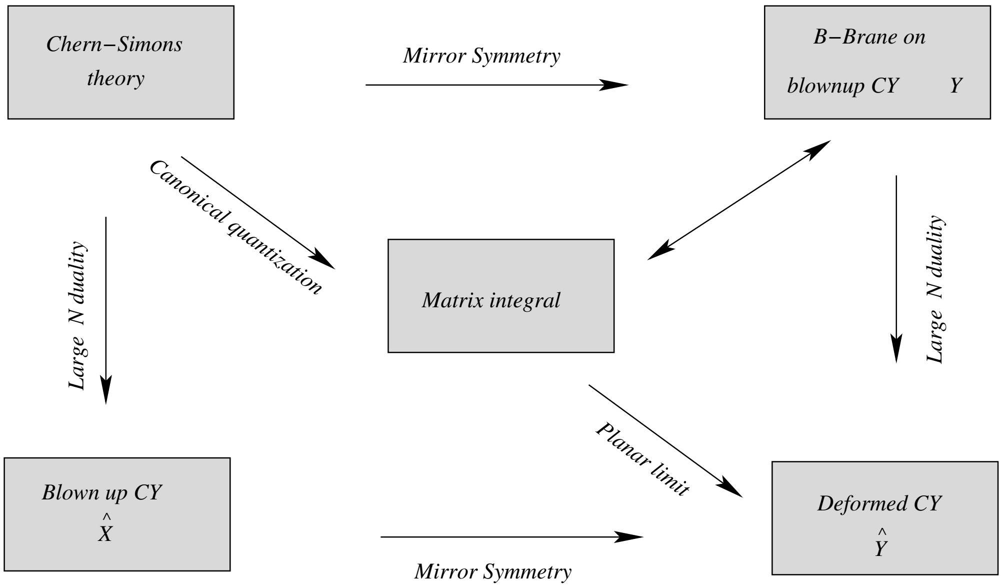


Figure 1: Interrelations of various topics covered in this paper.

from the pure Yang-Mills in terms of these multi-trace operators. Note that, upon lifting to M-theory, these theories give an effective description of $\mathcal{N} = 1$ compactifications of M-theory on certain G_2 manifolds.

The organization of this paper is as follows: in section 2, we review Chern-Simons theory and how it arises in the context of A-model topological strings. In particular we show that the matrix model expression of [1] for the partition function is already natural from the point of view of canonical quantization of Chern-Simons theory. In section 3, we present the mirror to the A-model geometries, following the ideas in [10]. We also analyze the topological theory describing B-branes in the mirror geometry in the spirit of [2], and we show that it reduces to a matrix model. This provides a mirror symmetry derivation of the Chern-Simons matrix models advanced in [1]. In section 4, we show that the standard planar limit analysis [11] of the matrix model leads to the mirror of the deformed conifold, showing in this way that the large-N limit of the Chern-Simons matrix model leads naturally to the mirror of the large-N transition proposed in [4]. In section 5, we extend the analysis to the case of lens spaces. We again give a mirror symmetry derivation of the corresponding matrix model describing Chern-Simons theory, and give a detailed comparison with standard results in Chern-Simons theory. Furthermore, we extend the large-N duality to the orbifolds of [4] by $\mathbb{Z}_p$. We do a detailed perturbative computation for $p = 2$, by rewriting the Chern-Simons matrix model for lens spaces as a hermitean matrix model. In section 6, we consider the closed string geometry which is the large-N dual of $T^*(\mathbf{S}^3/\mathbb{Z}_2)$, namely local $\mathbb{P}^1 \times \mathbb{P}^1$. We give a fairly complete description of the extended Kähler moduli space and we compute the F_g couplings by using the B-model Kodaira-Spencer theory of [12]. In order to test the large-N duality, we expand these coupling around the point in moduli space where both $\mathbb{P}^1$'s have vanishing quantum volume, and find perfect agreement with the results of matrix model/Chern-Simons perturbation theory. In section 7, we present some generalizations of the mirror symmetry derivation of the matrix model. In particular, we show how to include matter, making in this way contact with the results of [8, 9]. Finally, in section 8 we put our results in the context of type-IIA compactifications with spacetime filling branes, and we show that the resulting gauge theories include multi-trace operators that can be read off from the hermitean matrix model of section 5. Finally, the two appendices collect some useful results on computation of averages in the gaussian matrix model, and on the solution of the holomorphic anomaly equation.

2. Physics of the A-model and Chern-Simons theory

As shown in [3], if we wrap N D-branes on M in T^*M, the associated topological A-model is a U(N) Chern Simons theory on the three-manifold M

$$Z = \int \mathcal{D}A e^{S_{\mathrm{CS}}(A)} \tag{2.1}$$

where

$$S_{\mathrm{CS}}(A) = \frac{ik}{4\pi}\int_M \mathrm{Tr}\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right)$$

is the Chern-Simons action. The basic idea of this equivalence is as follows. The path-integral of the topological A-model localizes on holomorphic curves, and when there are D-branes, this means holomorphic curves with boundaries ending on them. In the T^*M geometry with D-branes wrapping M there are no honest holomorphic curves, however there are degenerate holomorphic curves that look like trivalent ribbon graphs and come from the boundaries of the moduli space. This leads to a field theory description in target space, which is equivalent to topological Chern-Simons theory. In this map, the level k would be naively related to the inverse of the string coupling constant g_s. However, quantum corrections shift this identification to

$$\frac{2\pi i}{k+N} = g_s\,.$$

The perturbative open-string expansion and Chern-Simons ribbon graph expansion around their classical vacua coincide.

In this paper we mainly consider M's that are T^2 fibered over an interval I. The fiber over a generic point in I is a T^2, but some (p,q) one-cycles of the T^2 degenerate at the end points. Alternatively, we can view M as obtained by gluing two solid tori T_L and T_R over the midpoint of the interval, up to an SL$(2,\mathbb{Z})$ transformation U that corresponds to a diffeomorphism identification of their boundaries. Let (p_L,q_L) be the cycle of the T^2 fiber that degenerates over the left half on M, and let (p_R,q_R) be the cycle that degenerates over the right half. The gluing matrix U can be written as

$$U = U_L^{-1}U_R, \tag{2.2}$$

where $U_{L,R} = \begin{pmatrix} p_{L,R} & s_{L,R} \\ q_{L,R} & t_{L,R}\end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$. (Clearly, U is unique up to a homeomorphism that changes the "framing" of three-manifold [13] and takes

$$V_{L,R} \to V_{L,R}\, T^{n_{L,R}} \tag{2.3}$$

JHEP02(2004)010

where T is a generator of $\mathrm{SL}(2,\mathbb{Z})$, $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. This is a consequence of the fact that there is no natural choice of the cycle that is finite on the solid torus. We will come back to this later.)

Consider M with an insertion of a Wilson line in representation R in T_L, and a Wilson line in representation R' in T_R along the one-cycles of the solid tori that are not filled in. The partition function is given by

$$Z(M; R, R') = \langle R|U|R'\rangle\,.$$

Above, $|R\rangle$ for example, corresponds to computing the path integral on the solid torus T_L. Moreover, it gives a state in the Hilbert space of Chern-Simons theory on T^2 on the boundary of T_L. The $\mathrm{SL}(2,\mathbb{Z})$ transformations of the boundary act as operators on this Hilbert space. The corresponding states and operators can be found by considering canonical quantization of Chern-Simons theory on $M = T^2 \times R$, following [14] (see also [15]). This allows one to solve the theory, and in particular to show that the theory is equivalent to a matrix model. Let us begin by briefly recalling [14].

By integrating over A_t where the time t corresponds to the R direction in $T^2 \times R$, the Chern-Simons path integral becomes

$$Z = \int \mathcal{D}A_u \mathcal{D}A_v\ \delta(F_{uv})\ \exp\left(\frac{k}{2\pi i}\int_M \mathrm{Tr}\, A_v \dot{A}_u\right). \tag{2.4}$$

The delta function localizes to A's which are flat connections on the T^2. As the fundamental group of the 2-torus is commutative, by a gauge transformation, we can set $A = u\ d\theta_u + v\ d\theta_v$ where u and v are holonomies of the gauge field along the $(1,0)$ and $(0,1)$ cycle of the T^2. Integrating out the unphysical degrees of freedom is rather subtle, but the main physical effect is to incorporate the shift of $k \to \hat{k} = k + N$. Thus, we can simply consider the naive quantization, with k replaced by $\hat{k}$ — the effective value of k is also what enters in the string coupling constant g_s.

We can now construct the operators representing the action of $\mathrm{SL}(2,\mathbb{Z})$ on the Hilbert space of T^2, by noting that u and v are conjugate variables, with

$$[u_i, v_j] = g_s \delta_{ij}\,.$$

The action of S and T operators on the T^2 implies that

$$T:\ u \to u + v\,,\quad v \to v\,;\qquad S:\ v \to u\,,\quad u \to -v\,,$$

and this suffices to determine them up to normalization [14]:

$$T = \eta_T\, \mathrm{e}^{-\mathrm{Tr}\, v^2/2g_s}\,;\qquad S = \eta_S\, \mathrm{e}^{-\mathrm{Tr}(u^2+v^2)/4\pi g_s}\,. \tag{2.5}$$

Suppose that the v-cycle of the T^2 is the one that is filled in. The wave function corresponding to the the path integral on the solid torus with insertion of a Wilson line in representation R along the cycle which is finite is given by

$$\langle v|R_v\rangle = \frac{1}{|\mathcal{W}|}\sum_{w\in\mathcal{W}} \epsilon(w)\delta(v + i g_s \omega(\alpha_R))\,. \tag{2.6}$$

The sum is over the elements w of the Weyl group where $\epsilon(w)$ is their signature. For $\mathrm{U}(N)$ the order of the Weyl group is $|\mathcal{W}| = N!$. Moreover, α_R is the highest weight vector of representation R, shifted by the Weyl vector $\rho = \frac{1}{2}\sum_{\alpha>0}\alpha$ with $\alpha > 0$ corresponding to positive roots. In particular, for the partition function without any insertions $\alpha_0 = \rho$.

In writing the wave function in equation (2.6) we do not divide by the full group of large gauge transformations on the T^2, but only by the Weyl group.[2] The path integral on the solid torus can be viewed as a path integral on an interval where v is frozen at the end-point where the v-circle is filled, and the large gauge transformations that shift v by $2\pi\alpha$ for α in the root lattice Λ_R are not a symmetry. In fact, generically the large gauge transformations are broken to the Weyl group by the operators in (2.5). This will be more transparent yet in the mirror B-model language.

Consider for example the partition function on a three-manifold M where $(p_L, q_L) = (0,1)$ and $(p_R, q_R) = (1,1)$, with no insertions. The gluing operator is $U = TST$, takes v to $u + v$, and leaves u invariant. In terms of u and v it is given by $U = \exp(\mathrm{Tr}\, u^2/2g_s)$, up to normalization. Correspondingly, we have

$$Z(M) = \langle 0_v | \exp(\mathrm{Tr}\, u^2/2g_s) | 0_v \rangle \,, \tag{2.7}$$

where $|0_v\rangle$ is the partition function on a solid torus with no insertions. By writing $|0_v\rangle$ in the u basis, we see that the theory can be described by a matrix model in terms of u, $\mathrm{e}^{iu} \in \mathrm{U}(N)$

$$Z = \frac{1}{\mathrm{vol}(\mathrm{U}(N))} \int d_H u \exp(\mathrm{Tr}\, u^2/2g_s)) \tag{2.8}$$

where $d_H u$ is the Haar measure on $\mathrm{U}(N)$. To show this, note that

$$\langle u | 0_v \rangle = \Delta_H(u) = \prod_{\alpha>0} 2\sin\left(\frac{\alpha \cdot u}{2}\right),$$

where we used Weyl denominator formula $\sum_{w\in\mathcal{W}} \epsilon(w) \exp(w(\rho)\cdot u) = \prod_{\alpha>0} 2\sinh(\frac{\alpha\cdot u}{2})$. Recall that the positive roots of $\mathrm{U}(N)$ are given by $\alpha_{ij} = e_i - e_j$, for $i < j$ where e_j form an orthonormal basis, and $\alpha_{ij} \cdot u = u_i - u_j$. On the other hand, it is a well known result that the Haar measure on $\mathrm{U}(N)$ becomes, when expressed in terms of the eigenvalues,

$$\frac{1}{\mathrm{vol}(\mathrm{U}(N))} \int d_H u = \frac{1}{|\mathcal{W}|} \int \prod_i du_i \, \Delta_H^2(u) \,, \tag{2.9}$$

upon integrating over angles. Therefore, (2.7) equals (2.8). Notice that, since we are not dividing by large gauge transformations, the integration region for the eigenvalues u_i is $\mathbb{R}^N$.

We can evaluate (2.8) explicitly by using the Weyl denominator formula to rewrite (2.9) as a gaussian integral. We find

$$Z = (-2\pi g_s)^{N/2} \eta_U \sum_{w\in\mathcal{W}} \epsilon(w) e^{\frac{g_s}{2}(\rho + w(\rho))^2}. \tag{2.10}$$

In the equation above, we denoted by η_U the normalization of the $U = TST$ operator which we have not fixed.

[2] In that the equation (2.6) differs from equation 4.12 of [14].

In [13], Chern-Simons theory was solved by relating the Hilbert space of Chern-Simons theory to the space of conformal blocks of WZW model. The action of $SL(2,\mathbb{Z})$ on the conformal blocks of WZW model allows one to read off the matrix elements of the operator corresponding to U. We will now show that the above matrix model formulation agrees with the known results for $U(N)_k$ WZW model on $\mathbf{S}^3$ with the corresponding framing. Namely, consider U corresponding to the $SL(2,\mathbb{Z})$ matrix

$$U = \begin{pmatrix} a & r \\ b & s \end{pmatrix}. \tag{2.11}$$

The path integral with Wilson lines in representation labeled by α_R, $\alpha_{R'}$ inserted parallel to the axis of the solid tori before the gluing with U are given by [16, 17]

$$\langle R|U|R'\rangle = c_U \sum_{n\in\Lambda_r/b\Lambda_r} \sum_{w\in\mathcal{W}} \epsilon(w) \exp\left\{\frac{i\pi}{\hat{k}b}(a\alpha_R^2 - 2\alpha_R\cdot(\hat{k}n + w(\alpha_{R'})) + s(\hat{k}n + w(\alpha_{R'}))^2)\right\}. \tag{2.12}$$

We recall that $\hat{k} = k + N$, and the coefficient c_U is given by

$$c_U = \frac{[i\,\mathrm{sign}(b)]^{N(N-1)/2}}{(\hat{k}|b|)^{N/2}} \exp\left[-\frac{i(N^2-1)\pi}{12}\Phi(U)\right], \tag{2.13}$$

that only depends on U and not on the Wilson-lines. Above, $\Phi(U)$ is the Rademacher function:

$$\Phi\left[\begin{pmatrix} a & r \\ b & s \end{pmatrix}\right] = \frac{a+s}{b} - 12s(a,b), \tag{2.14}$$

where $s(a,b)$ is the Dedekind sum

$$s(a,b) = \frac{1}{4b}\sum_{n=1}^{b-1} \cot\left(\frac{\pi n}{b}\right)\cot\left(\frac{\pi n a}{b}\right).$$

In particular, we see that the partition function on S^3 corresponding to $U = TST$ agrees with the expression we found above, provided we identify

$$\eta_{TST} = \frac{1}{(2\pi)^N} e^{-\frac{2\pi i(N^2-1)}{12}}.$$

We will make many further checks of this formalism in the following sections (In particular we will check that arbitrary matrix elements of U agree with (2.12).).

3. Mirror symmetry

3.1 Mirror pairs of geometries

As discussed above, Chern-Simons theory on a three-manifold M is the same as topological A-model string on T^*M. When M is a T^2 fibration over an interval, the geometry of $X = T^*M$ is rather simple. As shown in [10], X itself is a lagrangian $T^2\times\mathbb{R}$ fibration with base $\mathbb{R}^3$, and where one-cycles of the T^2 degenerate over lines in the base. Moreover the T^2 fiber of X and the fiber M can be identified. In the Calabi-Yau geometry, there is a natural choice of basis of $(1,0)$, $(0,1)$ cycles of the T^2 that fibers X, which is provided by the choice of complex structure on X. We can identify the one-cycles of the T^2 fiber that shrink over the left and the right sides of the interval with the shrinking 1-cycles of T_L and T_R. The diffeomorphism map U is the $\mathrm{SL}(2,\mathbb{Z})$ transformation that relates one of the shrinking cycles of the fiber of X to the other one. Moreover, while any path between the lines in $\mathbb{R}^3$ lifts to a three-manifold in X, the path of minimal length lifts to M.

For example, $X = T^*\mathbf{S}^3$ can be written as

$$xu + yv = \mu\,. \tag{3.1}$$

The T^2 fiber of X is visible from the fact that the equation is invariant under $\mathrm{U}(1)^2$ action where x, u are charged oppositely under the first and y, v under the second $\mathrm{U}(1)$. The minimal $\mathbf{S}^3$ embeds via $u = \bar{x}$ and $v = \bar{y}$,

$$|x|^2 + |y|^2 = \mu\,,$$

and if μ is real and positive this is a three-sphere. In view of the discussion above, we can regard this $\mathbf{S}^3$ as a real interval, together with the $(1,0)$ one-cycle of the T^2 fiber that corresponds to the phase of x and the $(0,1)$ cycle that is the phase of y. Alternatively, we have the gluing operator $U = S$. The $(1,0)$ and $(0,1)$ cycles degenerate over the $x = 0$ and $y = 0$ endpoints of the interval, respectively, and these are two copies of $\mathbb{C}^*$ in X — holomorphic cylinders $\mathbb{R} \times \mathbf{S}^1$.

As shown in [10], manifolds mirror to the above Calabi-Yau geometries can be obtained by deformation of the mirror duality proven in [18, 19]. We refer the reader to [10, 8] for the details of this and here simply state the result. Suppose M, viewed as a T^2 fibration, has (p_L, q_L) and (p_R, q_R) cycles of the T^2 which degenerate over the boundaries of the base interval. Correspondingly, X has two lines of degenerate fibers in the base. The mirror manifold of X, we will call it Y, is given by resolution of the following singularity

$$xy = P_L(u,v)P_R(u,v)\,, \tag{3.2}$$

where

$$P_L = e^{p_L u + q_L v} - 1\,, \qquad P_R = e^{p_R u + q_R v} - 1\,. \tag{3.3}$$

Above, u and v are $\mathbb{C}^*$ valued, so their imaginary parts are periodic, with period 2π. The resolution is by blowing up the locus $x = y = 0 = P_L = P_R$, by inserting a $\mathbb{P}^1$. If z, z' are coordinates on the $\mathbb{P}^1$, $z = 1/z'$ the resolution corresponds to covering X by two patches X_L and X_R given respectively by

$$(L) \quad xz = P_L\,, \qquad (R) \quad yz' = P_R\,,$$

in $x - z - u - v$ coordinates for X_L and in $y - z' - u - v$ space for X_R. The transition functions are obvious, relating e.g. $y = P_R z$.

JHEP02(2004)010

The minimal holomorphic $\mathbb{P}^1$ is where one is blowing up. This can be deformed to an $\mathbf{S}^2$ that is generally not holomorphic by letting x, y, u, v be arbitrary functions of $z, \bar{z}$ coordinates on the sphere, obeying above transition functions. However, the allowed deformations are not entirely arbitrary, as the equation of Y restricts the north pole of the $\mathbf{S}^2$ $(z = 0)$ and the south pole $(z' = 0)$ to lie at

$$(L) \quad P_L = 0\,, \qquad (R) \quad P_R = 0\,.$$

These deformations mirror the deformations of M in X. Topologically, M comes in a family of 3-submanifolds of X, by deforming the path in the base connecting the two lines arbitrarily, and the condition on the north and the south pole of the M to lie on the lines in base of X replaced by the above holomorphic constraint on the mirror two-spheres. This is natural in the view of the fact (which one can show using [18]) that the imaginary parts of u and v in the B-model are T-dual to the 1-cycles of the T^2 in the A-model [10, 8].

For example, the mirror of $T^*\mathbf{S}^3$ in (3.1) is given by blowup of

$$xy = (e^u - 1)(e^v - 1)$$

as described above. Mirror symmetry relates N D-branes wrapping the $\mathbf{S}^3$ in the A-model to N B-branes wrapping the $\mathbb{P}^1$ in the mirror geometry.[3]

3.2 The mirror B-model D-branes

In this section we consider B-branes, wrapping $\mathbb{P}^1$'s in the B-model geometries described above. We will show that the B-model theory is described by a matrix model, as in [2], albeit of a novel kind. By mirror symmetry, the B-branes on Y and the A-branes on X should give rise to the same theory. We will show that the matrix model describing the B-branes at hand is precisely the matrix model we arrived upon in section 2, by considering canonical quantization of Chern-Simons theory, and consequently the same matrix model as in [1].

In the simplest example, with $(p_{L,R}, q_{L,R}) = (0, 1)$ the manifold Y is given by blowing up

$$xy = (e^v - 1)^2\,, \qquad u\,.$$

This contains a family of $\mathbb{P}^1$'s parameterized by u, and is mirror to A-model geometry containing a family of $\mathbf{S}^2 \times \mathbf{S}^1$'s. Above, u and v are $\mathbb{C}^*$ valued, so their imaginary parts are periodic, with period 2π.[4]

We can choose to parameterize the normal directions to D-branes by v and u, and in terms of these, the action on the N D-branes wrapped on a $\mathbb{P}^1$ in this geometry is given by

$$S = \int_{\mathbb{P}^1} \mathrm{Tr}\, v^{(1)} \bar{D} u\,, \tag{3.4}$$

[3] The subtlety regarding the choice of framing of the three-manifold in X is related in part to performing global $\mathrm{SL}(2,\mathbb{Z})$ transformations of the T^2 fiber, which is a symmetry of the A-model theory. There is a similar subtlety in defining the B-model [20], and part of the framing ambiguity that can be traded for an $\mathrm{SL}(2,\mathbb{Z})$ transformation of the geometry corresponds in the B model to transformation that takes Y to $xy = (e^{u+mv} - 1)(e^v - 1)$.

[4] Note that if we did forget about compactness of v and of u the above geometry would be an A_1 ALE space times $\mathbb{C}$.

JHEP02(2004)010

where $v^{(1)} = v/zdz$ is a one-form on $\mathbb{P}^1$ valued in the Lie algebra of $\mathrm{U}(N)$, and $\bar{D} = \bar{\partial}+[A,\ ]$ for A a holomorphic $\mathrm{U}(N)$ connection on the $\mathbb{P}^1$. Note that v is a section of the trivial bundle on the $\mathbb{P}^1$ as $e^v = xz+1$ is globally defined on Y, and the same is true for u. The action is a non-abelian generalization of

$$S = \int_{B(\mathcal{C},\mathcal{C}_*)} \Omega$$

the action for a single D-brane on the $\mathbb{P}^1$ [2]. Above, $\Omega = \frac{dvdzdu}{z}$ is the holomorphic three-form on Y.[5] As a further check, note that the equations of motion corresponding to the action (3.4) have solutions which agree with the geometric picture. That is

$$\bar{D}u = 0 = \bar{D}(v/zdz)\,,$$

is solved by u an arbitrary constant on the D-brane, and moreover the v equation of motion requires $v \sim z$ near the north pole $z = 0$ and $v \sim z'$ near the south pole $z' = 0$, and is therefore zero throughout. In terms of the path-integral, the action localizes on the paths for which v vanishes on the north and the south poles of the sphere, and the equations of the blowup imply this as well.

Note that (3.4) is the same as the action of Chern-Simons in the temporal gauge, provided we identify the holonomies around the two 1-cycles of the T^2 in Chern-Simons. In fact mirror symmetry provides this identification naturally! Since the T^2 in the B-model, corresponding to the imaginary parts u, v variables in Y being compact, is mirror to the T^2 that fibers X, the identification of variables above follows simply by applying T-duality on the D-branes (To be precise, in comparing to (2.4) one should also replace the $\mathbb{P}^1$ by a cylinder, by replacing $dz/z = d\rho$, where the cylinder is parameterized by ρ.).

For more general three-manifolds equations (3.2), (3.3), the north and the south pole of the D-brane are constrained to live on $p_L u + q_L v = 0$ and $p_R u + q_R v = 0$. We can think of the theory on the D-brane as obtained by gluing together two halves of $\mathbb{P}^1$'s [2]. The action on both halves is the same, as the holomorphic three-form Ω is the same, but there is a non-trivial map between the two boundaries. That is, writing the partition function on the $\mathbb{P}^1$ as $Z = \langle \rho_L | \rho_R \rangle$, the states $|\rho_{L,R}\rangle$ are obtained by evaluating the path integral over the north and the south cap of the $\mathbb{P}^1$. In the present context, these correspond to imposing the boundary conditions $p_{L,R}u + q_{L,R}v = 0$, classically, so we can denote

$$|\rho_{L,R}\rangle = |\,0_{p_{L,R}u+q_{L,R}v}\rangle\,,$$

therefore

$$Z = \langle 0_{p_L u+q_L v}\,|\,0_{p_R u+q_R v}\rangle\,.$$

Note that u and v are conjugate variables in the lagrangian, so if we know $|0_v\rangle$, the state corresponding to $|\,0_{pu+qv}\rangle$ is related to it by an operator U

$$|0_{pu+qv}\rangle = U_{(p,q)}\,|0_v\rangle\,,$$

[5]Here, $v(z,\bar{z})$ and $u(z,\bar{z})$ are viewed as maps deforming the holomorphic curve $\mathcal{C}_*$ to a nearby curve $\mathcal{C}$ which is not holomorphic, and $B(\mathcal{C},\mathcal{C}_*)$ is the 3-chain interpolating between them. Evaluated for an infinitesimal deformation along the v direction, this gives the action (3.4) for a single D-brane.

such that

$$U_{(p,q)} v U_{(p,q)}^{-1} = pu + qv\,,$$

as discussed above in the Chern-Simons context.

Moreover, in the present context, the effective gluing operator, $U = U^{-1}_{(p_L,q_L)} U_{(p_R,q_R)}$, should be naturally related to the superpotential W of the theory. Namely, the operator U encodes difference of boundary conditions on the north and the south poles of the $\mathbb{P}^1$ which is what makes the supersymmetric vacua in the generic geometry (3.2) isolated. In turn, this is precisely what the superpotential W encodes. As an example, consider $v = 0$ as the boundary conditions on the left half of the $\mathbb{P}^1$, and $u + v = 0$ on the right, corresponding to a B-brane on

$$xz = (e^v - 1)(e^{v+u} - 1)\,. \tag{3.5}$$

Then, $U = \exp(\frac{1}{2g_s}\int_{\mathbb{P}^1} \omega \operatorname{Tr} u^2)$, where ω is a $(1,1)$ form on $\mathbb{P}^1$ of unit volume. The blowup of the manifold in (3.5) corresponds to mirror of $T^*\mathbf{S}^3$ with non-trivial framing that we studied in detail in section 2. In fact,

$$U = \exp\left(\frac{1}{g_s}\int_{\mathbb{P}^1} \omega \operatorname{Tr} W(u)\right).$$

Namely, we can compute the superpotential by considering a deformation of the holomorphic 2-sphere $\mathcal{C}_* = \mathbb{P}^1$ by giving u a constant value on the $\mathbb{P}^1$. This deforms $\mathcal{C}_*$ to a nearby sphere $\mathcal{C}(u)$ which is not holomorphic. Then the superpotential is given by [21, 22]

$$W(u) = \int_{B(\mathcal{C}(u),\mathcal{C}_*)} \Omega\,.$$

We find[6]

$$W(u) = \frac{1}{2}u^2\,, \tag{3.6}$$

as claimed above.

The state $|0_v\rangle$ can be found as follows. In the context of a single D-brane, this is a simple δ-function at $v = 0$ since we have a non-interacting theory. That is, we have $|0_v\rangle = \int \mathcal{D}v\mathcal{D}u \exp(\frac{1}{g_s}\int_{\frac{1}{2}\mathbb{P}^1} v^{(1)}\bar{\partial}u)$ which integrating over v reduces to zero modes of u and so

$$|0_v\rangle = \int du |u\rangle\,,$$

which is the same as in [2]. More generally, for N D-branes on the $\mathbb{P}^1$ u, v are promoted to matrices in the U(N) Lie algebra, and this will lead to non-trivial measure factors in the path integral written in terms of eigenvalues.

[6] As explained in more detail in [10] one can simplify the calculation by using independence of the three-form periods on blowing up the geometry, which is a Kähler deformation, and compute the integral in the singular geometry. At fixed value of u, $\int dxdv/x$ integral computes the holomorphic volume of the special lagrangian $\mathbf{S}^2$ in the two-fold fiber, and this is u.

Note that since u, v are periodic in the geometry, the natural measure for N D-branes is not the hermitean matrix measure as in [2], but the unitary matrix measure, corresponding to a hermitean matrix with compact eigenvalues. That is, for example in the B-model mirror to $\mathbf{S}^2 \times \mathbf{S}^1$ we have

$$\langle 0_v | 0_v \rangle = \frac{1}{\mathrm{vol}(\mathrm{U}(N))} \int d_H u = \frac{1}{|\mathcal{W}|} \int \prod_i du_i \; \Delta_H^2(u) \,, \tag{3.7}$$

where in the second equality we integrated over the angular variables of matrix u to get

$$\Delta_H(u) = \prod_{i<j} 2 \sin\left(\frac{u_i - u_j}{2}\right).$$

This differs from the hermitean matrix measure $\Delta(u) = \prod_{i<j}(u_i - u_j)$, and $\Delta_H(u)$ can be interpreted as a hermitean measure in which we include the images of the D-brane [23], i.e.

$$\Delta_H(u) \sim \prod_n \prod_{i<j} (u_i - u_j + 2\pi n) \,.$$

By taking the square root of (3.7), we find that

$$|0_v\rangle = \frac{1}{|\mathcal{W}|^{1/2}} \int \prod_i du_i \Delta_H(u) |u\rangle,$$

where $\mathbf{u}_i|u\rangle = u_i|u\rangle$ is the eigenstate of operator u.

It is important to note that while u and v are periodic in Y, the physics of the B-branes in these models generally does not have any periodicity, because the boundary conditions imposed generally break this. One can see this already by considering a single D-brane in (3.5). Taking u to $u + 2\pi$ the $\mathbf{S}^2$ winds around the v cylinder once: the south pole is at $v = 0$ and the north pole at $v = -u = -2\pi$, and consequently the D-brane does not come back to itself. Alternatively, the superpotential is not periodic in u, and this corresponds to the fact that the tension of the D-brane increases in going around. Consequently, the range of all integrations is non-compact.

The example of a B-brane on (3.5) consequently gives a hermitean matrix model, but with unitary measure

$$Z_{TST} = \langle 0_v | 0_{u+v} \rangle = \frac{1}{\mathrm{vol}(\mathrm{U}(N))} \int d_H u \; e^{\frac{1}{2g_s} \mathrm{Tr}\, u^2} . \tag{3.8}$$

More general examples can be constructed along similar lines, and we will see some of them in the following sections.

4. Planar limit

In [4] it was shown that holes in the topological open string amplitudes for N D-branes on $\mathbf{S}^3$ in $X = T^*\mathbf{S}^3$ can be summed up, genus by genus. The resulting closed string amplitudes coincided with that of closed topological A-model on $\hat{X} = \mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$. In the

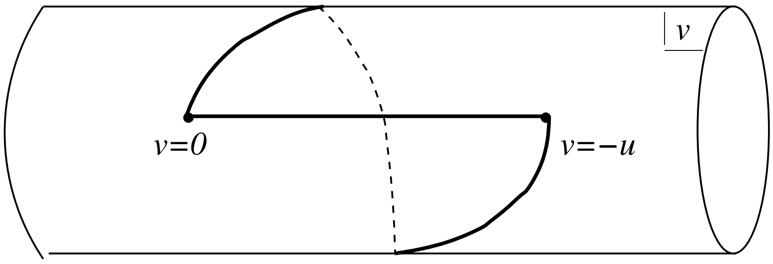


Figure 2: The B-brane projected to v cylinder corresponds to a path between $v = 0$ and $v = -u$. Because the boundary conditions on the two endpoints are different, going around $u \to u = 2\pi$, the B-brane does not come back to itself.

previous sections we showed that the Chern-Simons theory on $\mathbf{S}^3$ can be rewritten as a matrix model

$$Z = \frac{1}{\mathrm{vol}(\mathrm{U}(N))} \int d_H u \ \exp\left(\frac{1}{2g_s} \mathrm{Tr}\, u^2\right) \tag{4.1}$$

that naturally arises as the theory on the mirror B-model D-branes. In this section we want to show that the matrix model is solvable in the planar limit, and that the geometry which emerges is precisely that of the mirror $\hat{Y}$ of $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$.

As discussed before, after integrating over angular variables, (4.1) can be written as

$$\int \prod_i du_i \Delta_H(u)^2 \exp\left(-\frac{1}{2g_s} \sum_i u_i^2\right).$$

Note that in writing the above integral we have made a choice of the integration contour which amounts to $u_j \to i u_j$, and then $\Delta_H(u) = \prod_{i<j} 2\sinh(\frac{u_i - u_j}{2})$. In the large-$N$ limit, the integral is localized to the saddle point,

$$\frac{1}{g_s} u_i = \sum_{i \neq j} \coth\left(\frac{u_i - u_j}{2}\right), \tag{4.2}$$

and we can replace the discrete set of eigenvalues u_i by a continuous function $u(s)$. The sum in (4.2) becomes an integral and we find

$$-\frac{1}{t} u(s_0) = \mathrm{P} \int_0^1 ds \coth(u(s) - u(s_0)),$$

where P denotes the principal value, and $t = N g_s$ is the 't Hooft parameter. To solve the above equation we follow [11] and introduce a density of eigenvalues $\rho(u)$. We now change variables from u to $U = e^u$. The density satisfies $\rho(U) dU/U = ds$, and from $\int_0^1 ds$ we have

$$\int_a^b \rho(U) \frac{dU}{U} = 1. \tag{4.3}$$

In terms of u the above equation (4.2) is

$$-\frac{1}{2t} \log(U e^{-t}) = \mathrm{P} \int_a^b \frac{\rho(U')}{U' - U} dU', \tag{4.4}$$

so solving (4.2) is equivalent to solving for the density of eigenvalues u that satisfies equations (4.3), (4.4). The solution of (4.4) is now standard. Namely, we can define a function $v(U)$ (usually called the resolvent) by

$$v(U) = t\int_a^b \frac{\rho(U')}{U'-U}dU'\,,$$

and then the conditions on u are equivalent to asking that (i) v is analytic in the complex U plane, cut along an interval (a,b); (ii) it decays at infinity as $1/U$; (iii) the period of v around the cut is $2\pi it$; (iv) as U approaches the interval, $v(U\pm i\epsilon) = -\frac{1}{2}\log(Ue^{-t}) \pm \pi it\rho(U)$.

These conditions suffice to completely fix $v(U)$, to be

$$v = \log\left[\frac{1+e^{-u}+\sqrt{(1+e^{-u})^2-4e^{-u+t}}}{2}\right].$$

The zeros of the square root in the above expression correspond to the endpoints of the cut. Alternatively, v and u are functions on the Riemann surface

$$(e^v-1)(e^{v+u}-1)+e^t-1=0\,, \tag{4.5}$$

and moreover there is a one-form vdu whose periods on the Riemann surface satisfy special geometry:

$$t = \frac{1}{2\pi i}\int_A vdu\,, \tag{4.6}$$

and

$$\partial_t F_0 = \frac{1}{2\pi i}\int_B vdu\,, \tag{4.7}$$

where A-cycle corresponds to integrating around the cut, and the B-cycle corresponds to an integral from the endpoint of the cut to some cut-off point at large u.

Note that on the one hand, the Riemann surface (4.5) is the non-trivial part of the geometry

$$xz = (e^v-1)(e^{v+u}-1)+e^t-1\,, \tag{4.8}$$

that arises by geometric transition that blows down the $\mathbb{P}^1$ in (3.5) and deforms it by giving t a non-zero value. On the other hand, the equation (4.8) precisely describes the mirror of $\hat{X} = O(-1)\oplus O(-1)\to\mathbb{P}^1$ [18], where the size of the $\mathbb{P}^1$ is mirror to t in (4.8).

In this sense, we have derived the mirror of $\hat{X}$ by showing the equivalence of the open string A- and the B-model, and taking the large-N limit of both. It can be shown by explicit calculation that the function F_0 in (4.7) precisely agrees with the genus zero partition function of the A-model on $\hat{X}$ and the sum over the planar diagrams in U(N) Chern-Simons theory.

5. Lens spaces

In this section we consider a generalization of the above results where we replace $\mathbf{S}^3$ with the lens spaces $M_p = \mathbf{S}^3/\mathbb{Z}_p$, where $\mathbb{Z}_p$ acts on $\mathbf{S}^3$ as

$$|x|^2 + |y|^2 = 1\,, \qquad (x, y) \sim \exp\left(\frac{2i\pi}{p}\right)(x, y)\,. \tag{5.1}$$

We can think of this as obtained by gluing two solid 2-tori along their boundaries after performing the $\mathrm{SL}(2,\mathbb{Z})$ transformation,

$$U_p = \begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix}. \tag{5.2}$$

To see that, consider an $\mathbf{S}^3$ which, as explained above, is a T^2 fibration over an interval, where the cycles of the T^2 are generated by phases of x, y. If the complex structure of the T^2 corresponding to $\mathbf{S}^3$ it is τ, then an $\mathrm{SL}(2,\mathbb{Z})$ transformation that takes this T^2 to a T^2 with $(1,0)$ and $(1,p)$ cycles vanishing over the endpoints will take τ to $\tau' = \frac{\tau+1}{p}$. But the T^2 with the new complex structure is precisely a quotient of the original one by the $\mathbb{Z}_p$ action specified in (5.1).

Wrapping N D-branes on M_p in T^*M_p, the topological A-model is $\mathrm{U}(N)$ Chern-Simons theory on M_p. The critical points of the CS action are flat connections, which are classified by embeddings of the first fundamental group in $\mathrm{U}(N)$. Since $\mathbb{Z}_p$ acts freely on $\mathbf{S}^3$, we have that $\pi_1(M_p) = \mathbb{Z}_p$. Therefore, on M_p there are $\mathbb{Z}_p$ discrete flat connections we can turn on. A choice of a flat connection breaks the gauge group $\mathrm{U}(N) \to \mathrm{U}(N_1) \times \ldots \times \mathrm{U}(N_p)$, and leads to a choice of vacuum of the theory. The full partition function of Chern-Simons theory on a compact manifold involves summing over all the flat connections, and in fact the nonperturbative answer that can be obtained from the relation with WZW theory [13] gives such a sum. However, for our applications we are interested in Chern-Simons theory expanded around a particular vacuum, so in evaluating Chern-Simons amplitudes the prescription is not to sum over different flat connections. Namely, although Chern-Simons theory lives in a compact space, in our applications D-branes are wrapping not M_p but $M_p \times \mathbb{R}^4$, corresponding to type-IIA compactification on $T^*M_p \times \mathbb{R}^4$.

In this section we will first show how to generalize the B-matrix model to the case of lens space, and we will explicitly show that it agrees with the direct computation using the standard techniques in CS. We will also discuss the large-N transition for CS on lens spaces, and we will introduce a hermitean multi-matrix model for CS that captures the contribution of a given vacuum.

5.1 B-model matrix model

From the discussion in previous sections, the mirror of T^*M_p is given by blowing up

$$xy = (e^v - 1)(e^{v+pu} - 1)\,,$$

corresponding to the fact that in the A-model, there are two lines in the base $\mathbb{R}^3$ over which the $(0,1)$ and $(p,1)$ cycles of the torus degenerate. The resolved geometry,

$$xz = e^v - 1\,, \qquad u$$

in the $z-$patch and

$$yz' = e^{pu+v} - 1\,,$$

in the $z'-$patch, $z = 1/z'$. There are p holomorphic $\mathbb{P}^1$'s at $v = 0 = pu$, i.e. at

$$(u, v) = \left(\frac{2\pi i k}{p}, 0\right), \qquad k = 0, \dots, p-1\,.$$

Wrapping N D-branes in this geometry, one has to decide how to distribute the N D-branes among the p vacua. This we can see it at a quantitative level as well. By a trivial generalization of (3.6), it is easy to see that the theory on the wrapped D-branes has a superpotential $W_p(u)$, where

$$W_p(u) = \frac{pu^2}{2}\,,$$

and this has p vacua as claimed. The B-model path integral, as explained in section 3, is

$$Z = \langle 0_v | 0_{v+pu} \rangle = \frac{1}{\mathrm{vol}(\mathrm{U}(N))} \int d_H u \, \mathrm{e}^{-\frac{1}{g_s} \mathrm{Tr}\, W_p(u)}, \tag{5.3}$$

since

$$\exp\left(\frac{1}{g_s} \mathrm{Tr}\, W_p(u)\right) : (u, v) \to (u, v + pu)\,.$$

Distributing the N branes among the p different vacua corresponds, in the matrix model, to distributing the N eigenvalues among the different critical points, and also to the choice of a flat connection in the Chern-Simons theory.

Consider now the path integral around the critical point where N_k eigenvalues are at $u_j = 2\pi i(j-1)/p,\ \ j = 1, \dots p$, and the gauge group is broken as $\mathrm{U}(N) \to \mathrm{U}(N_1) \times \cdots \times \mathrm{U}(N_p)$. In the eigenvalue basis, the matrix model reads:

$$Z = \int \prod_{j=1}^{p} \frac{d^{N_j} u^{(j)}}{N_j!} \Delta_H(u^{(j)})^2 \prod_{j<k} \Delta_H\left(u^{(j)}, u^{(k)}\right)^2 \exp\left\{-\sum_j \mathrm{Tr}\, p(u^{(j)})^2/2g_s\right\} \tag{5.4}$$

where we have denoted by $u^{(j)}$ the set of N_i eigenvalues sitting at $2\pi i(j-1)/p$, and

$$\begin{aligned}
\Delta_H(u^{(j)}) &= \prod_{m<n} 2\sinh\left(\frac{u_m^{(j)} - u_n^{(j)}}{2}\right), \\
\Delta_H(u^{(j)}, u^{(k)}) &= \prod_{m,n} 2\sinh\left(\frac{u_m^{(j)} - u_n^{(k)} + d_{jk}}{2}\right),
\end{aligned}$$

where $d_{jk} = 2\pi i(j-k)/p$. In other words, there is an effective interaction between D-branes at different vacua. This can be thought of as coming from integrating out at one loop the massive string states stretched between the branes.

5.2 Chern-Simons theory on $\mathbf{S}^3/\mathbb{Z}_p$

In this subsection we show that there is an exact agreement between the topological B-model and the Chern-Simons answer, as expected. To do that, we will rewrite the matrix model (5.4) in the eigenvalue basis in a slightly different way. Consider the integral

$$\int \prod_{k=1}^{N} du_k \, \mathrm{e}^{-\sum_j u_j^2/2\hat{g}_s - \hat{k}\sum_j n_j u_j} \prod_{j<k} \left(2\sinh \frac{u_j - u_k}{2} \right)^2 , \tag{5.5}$$

where the effective coupling constant $\hat{g}_s$ is given by

$$\hat{g}_s = \frac{2\pi i}{p\hat{k}} . \tag{5.6}$$

In (5.5), we have also introduced a vector n of N integer numbers $0 \leq n_j \leq p-1$ that label at which critical point is the eigenvalue u_j. These integers label the choice of vacuum $\mathrm{U}(N) \to \mathrm{U}(N_1) \times \cdots \times \mathrm{U}(N_p)$ as follows: N_k is the number of n_j's equal to $k-1$. Notice that there is not a one-to-one correspondence between the n_j's and the different vacua, since any Weyl permutation of the n_j gives the same N_k's. Therefore, there are in total

$$\frac{N!}{\prod_{k=1}^{p} N_k!} \tag{5.7}$$

configurations of n_j's that correspond to the same vacuum. Notice however that (5.5) is manifestly invariant under permutations of the n_j's, so we can just pick any one of them. If we now change variables in (5.5) by $u_j \to u_j + \hat{k}\hat{g}_s n_j$, we reproduce (5.4).

According to our general results, the integral (5.5) must be the contribution of the flat connection labeled by $\{N_k\}_k$ to the partition function of CS theory on M_p. This follows indeed from [1], but in the case of lens spaces one can prove it in a very simple way. After using Weyl's denominator formula, the integral (5.5) becomes just a gaussian, and it can be computed to give (up to overall constants)

$$\frac{1}{|\mathcal{W}|} \sum_{w', w'' \in \mathcal{W}} \epsilon(w')\epsilon(w'') \exp\left\{ \frac{i\pi}{\hat{k}p} \left(w'(\rho) - \hat{k}n - w''(\rho) \right)^2 \right\} . \tag{5.8}$$

If we now sum (5.8) over all possible n, we obtain the following expression

$$\sum_{n \in \mathbb{Z}^N/p\mathbb{Z}^N} \sum_{w \in \mathcal{W}} \epsilon(w) \exp\left\{ \frac{i\pi}{\hat{k}p} \left(\rho^2 - 2\rho \cdot (\hat{k}n + w(\rho)) + (\hat{k}n + w(\rho))^2 \right) \right\} , \tag{5.9}$$

To see this, notice that the lattice $\mathbb{Z}^N/p\mathbb{Z}^N$ in (5.9) is invariant under Weyl permutations, therefore we can sum over all possible permutations of n and divide by the order of the Weyl group $|\mathcal{W}|$. In this way we obtain (5.8), summed over all n. In this way, we have rederived the matrix element (2.12) when U_p is the $\mathrm{SL}(2,\mathbb{Z})$ element (5.2). Since this matrix element is the partition function of CS theory on the lens space M_p, we have shown

JHEP02(2004)010

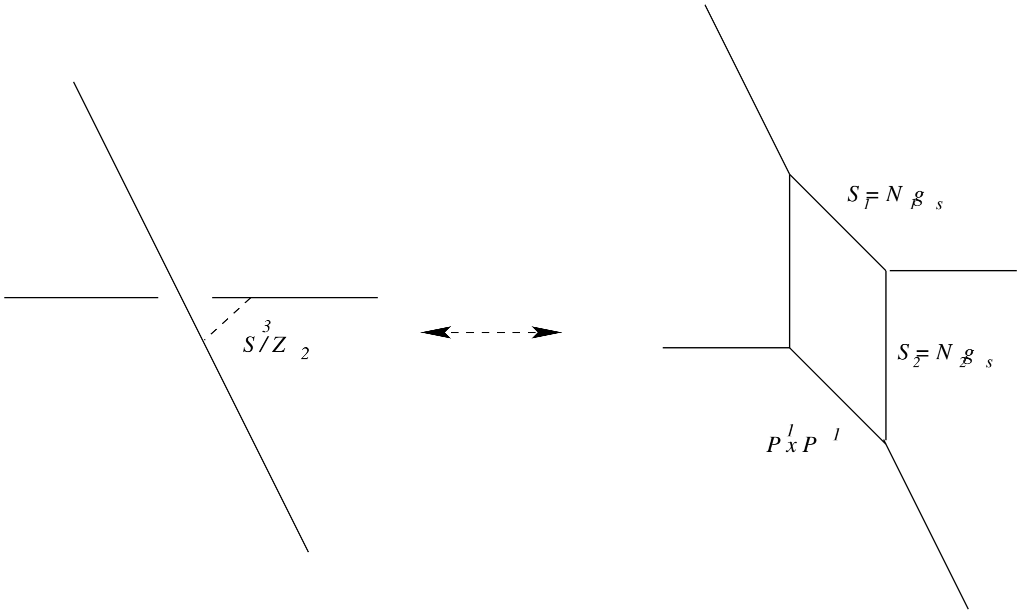


Figure 3: The figure depicts a geometric transition between $T^*(\mathbf{S}^3/\mathbb{Z}_2)$ and $O(-K) \to \mathbb{P}^1 \times \mathbb{P}^1$. With N D-branes on $\mathbf{S}^3/\mathbb{Z}_2$ and gauge group broken to $\mathrm{U}(N_1) \times \mathrm{U}(N_2)$ with $N = N_1 + N_2$, the geometric transition is a large-N duality and the BPS sizes $S_{1,2}$ of two $\mathbb{P}^1$'s are identified with the t'Hooft parameters $S_i = N_i g_s$.

that the integral (5.4) gives precisely the contribution of the flat connection labeled by $\{N_k\}_k$ to the CS partition function. After including all the overall factors carefully, one finds that the precise expression of the full partition function in the canonical framing is

$$\sum_n \frac{\mathrm{e}^{-\frac{\hat{g}_s}{12}N(N^2-1)}}{N!} \int \prod_{i=1}^{N} \frac{du_i}{2\pi}\, \mathrm{e}^{-\sum_i u_i^2/2\hat{g}_s - \hat{k}\sum_i n_i u_i} \prod_{i<j} \left(2\sinh\frac{u_i-u_j}{2}\right)^2 . \tag{5.10}$$

5.3 Large N duality for lens spaces

In [4], the large-N limit of topological open strings on $T^*\mathbf{S}^3$ was shown to be given by closed topological strings on the resolved conifold $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$. There is a natural question of what is the large-N limit when we replace $\mathbf{S}^3$ with $\mathbf{S}^3/\mathbb{Z}_p$. The answer for this, generalizing [4], is as follows. For definiteness, consider first $p = 2$. As is familiar, $X = T^*(\mathbf{S}^3/\mathbb{Z}_2)$ has a geometric transition where $\mathbf{S}^3/\mathbb{Z}_2$ is replaced by $F_0 = \mathbb{P}^1 \times \mathbb{P}^1$. The total geometry is a cone over this, more precisely it is $\hat{X} = O(-K) \to F_0$.

That such transition is allowed is easy to see in the language of (p, q) five-branes, see figure 3. One may then expect that the large-N limit of N D-branes on $\mathbf{S}^3/\mathbb{Z}_2$ is a closed string theory on $\hat{X}$. For general p, the dual geometry is an A_{p-1} fibration over $\mathbb{P}^1$, with p complexified Kähler classes corresponding to the sizes of the p different $\mathbb{P}^1$'s.

In order to make precise the implications of this large-N transition, we need an identification of the parameters between the two theories. On the open string side we have a choice of the numbers N_k of D-branes to place in the p different vacua, and we would expect that these choices correspond to changing the sizes of the p $\mathbb{P}^1$'s. The natural identification is as follows.

Recall that in the open string theory the large-N expansion is a weak coupling expansion in g_s. The open string free-energy is of the form

$$F = F^{\rm nonpert} + F^{\rm pert}\,,$$

where

$$F^{\rm pert} = \sum_{g=0}^{\infty} F^{\rm pert}_{g,h}(N_k) g_s^{2g-2+h}\,.$$

The N_k dependence in $F_g^{\rm pert}$ comes from tracing over the Chan-Paton indices of Riemann surfaces with holes. This expansion is nothing but the Feynman-diagram expansion of the CS path integral in the background of a flat connection given by the N_k's. Notice that $F^{\rm pert}_{g,h}(N_k)$ has in fact the structure

$$F^{\rm pert}_{g,h}(N_k) = \sum_{h_1+\cdots+h_p=h} F^{\rm pert}_{g,h_1,\ldots,h_p} N_1^{h_1}\cdots N_p^{h_p}\,, \tag{5.11}$$

i.e. it is a homogeneous polynomial in N_k of degree h. The non-perturbative piece, in contrast to the model dependence of $F^{\rm pert}$, has a universal behavior. From the open string/CS perspective, this comes from the measure of the path integral — basically the volume of the unbroken gauge group G [5].

$$F^{\rm nonpert} \sim -\log({\rm vol}(G))\,.$$

In our case $G = {\rm U}(N_1)\times\cdots\times{\rm U}(N_p)$, and the explicit expression of $F^{\rm nonpert}$ can be easily obtained from the asymptotic expansion

$$\log({\rm vol}({\rm U}(N))) = -\frac{N^2}{2}\left(\log(N)-\frac{3}{2}\right)+\frac{1}{12}\log N+\frac{1}{2}N^2\log 2\pi - \tag{5.12}$$

$$-\sum_{g=2}^{\infty}\frac{B_{2g}}{2g(2g-2)}N^{2-2g}\,. \tag{5.13}$$

In order to identify the parameters in the open and the closed string side, consider the genus zero piece of the nonperturbative part of the free energy:

$$F^{\rm nonpert}_{g=0} = \frac{1}{2}\sum_{i=1}^{p}(g_sN_i)^2\log(g_sN_i)\,, \tag{5.14}$$

where g_sN_i is the 't Hooft coupling. This universal behavior strongly suggests the following. The genus zero topological closed string amplitudes with the above form are well known to arise by integrating out nearly massless charged particles of mass g_sN_i, since (5.14) is basically a contribution of BPS D-branes at one loop to the $\mathcal{N}=2$ prepotential. Therefore, one may naively identify the 't Hooft parameters g_sN_i with the flat coordinates S_i measuring the BPS sizes of the p $\mathbb{P}^1$'s:

$$S_i = g_sN_i\,,\qquad i=1,\ldots,p\,. \tag{5.15}$$

Notice that in this picture the perturbative piece of the open string free energy

$$F_g^{\rm pert}(S_i) = \sum_{g,h_1,\ldots,h_p} F_{g,h_1,\ldots,h_p}^{\rm pert}(g_sN_1)^{h_1}\cdots(g_sN_p)^{h_p}$$

which can be computed in ordinary Chern-Simons perturbation theory, is the regular part of the $F_g^{\rm closed}$ coupling for the closed string dual geometry, expanded in terms of flat coordinates S_i around the point in moduli space where the $\mathbb{P}^1$'s have vanishing size. We will refer to this point as the orbifold point, although in contrast to orbifold points in other geometries, like local $\mathbb{P}^2$, we have a singular behavior of the prepotential captured by $F^{\rm nonpert}$. We will show below that the naive expectation (5.15) is correct, by comparing the perturbative expansion in the open side with the expansion of $F_g^{\rm closed}$ computed in the B model around the orbifold point.

5.4 (Hermitean) Matrix model for Chern-Simons on lens spaces

In order to test the large-N duality for lens spaces in the way that we just suggested, we have to compute $F_{g,h}$ in the open string/CS side. To do this the equivalence between CS theory and matrix models turns out to be very useful. As pointed out in [1], one can regard the CS matrix model as a "deformation" of the usual hermitean gaussian model, where the deformation is due to the appearance of $\prod_{i<j}(2\sinh((u_i-u_j)/2))^2$ instead of the usual Vandermonde determinant $\prod_{i<j}(u_i-u_j)^2$, therefore one can systematically compute the perturbative expansion of the CS theory in terms of perturbation theory of the gauged matrix model around the gaussian point.

We will in fact write a *hermitean* matrix model underlying (5.10). Let us first consider the contribution due to the trivial connection, i.e. let us consider the integral in (5.10) with $n=0$. We now do the following trick. As in [1], we write

$$\prod_{i<j}\left(2\sinh\frac{u_i-u_j}{2}\right)^2 = \Delta^2(u)f(u)\,. \tag{5.16}$$

In this equation $\Delta(u)=\prod_{i<j}(u_i-u_j)$ is the usual Vandermonde determinant, and the function $f(u)$ is given by

$$f(u) = \exp\left(\sum_{k=1}^{\infty} a_k\sigma_k(u)\right), \tag{5.17}$$

where

$$\begin{aligned}\sigma_k(u) &= \sum_{i<j}(u_i-u_j)^{2k}\,,\\ a_k &= \frac{B_{2k}}{k(2k)!}\end{aligned} \tag{5.18}$$

and B_{2k} are the Bernoulli numbers. a_k are simply the coefficients in the expansion of $\log(2\sinh(x/2)/x)$. The $\sigma_k(u)$ are symmetric polynomials in the u_i's, therefore can be written in terms of Newton polynomials

$$P_j(u) = \sum_{i=1}^{N} u_i^j\,, \tag{5.19}$$

as follows:

$$\sigma_n(u) = NP_{2n}(u) + \frac{1}{2}\sum_{s=1}^{2n-1}(-1)^s\binom{2n}{s}P_s(u)P_{2n-s}(u)\,. \tag{5.20}$$

We then write the integral as:

$$\frac{\mathrm{e}^{-\frac{\hat{g}_s}{12}N(N^2-1)}}{N!}\int\prod_{i=1}^{N}\frac{du_i}{2\pi}\Delta^2(u)\,\exp\left(-\sum_i\frac{u_i^2}{2\hat{g}_s}+\sum_{k=1}^{\infty}a_k\sigma_k(u)\right). \tag{5.21}$$

Now we notice that the Newton polynomials $P_j(u)$ are just $\mathrm{Tr}\,M^j$, where M is a hermitean matrix which has been gauge-fixed to the diagonal form $\mathrm{diag}(u_1,\ldots,u_N)$. Therefore the above integral is (up to the prefactor $\mathrm{e}^{-\frac{\hat{g}_s}{12}N(N^2-1)}$) the gauge-fixed version of the hermitean matrix model

$$\frac{1}{\mathrm{vol}(\mathrm{U}(N))}\int dM\exp\left\{-\frac{1}{2\hat{g}_s}\,\mathrm{Tr}\,M^2+V(M)\right\}, \tag{5.22}$$

where

$$V(M) = \frac{1}{2}\sum_{k=1}^{\infty}a_k\sum_{s=0}^{2k}(-1)^s\binom{2k}{s}\,\mathrm{Tr}\,M^s\,\mathrm{Tr}\,M^{2k-s}. \tag{5.23}$$

Here we used the expression for the hermitean measure (see for example the second appendix in [24])

$$\frac{1}{\mathrm{vol}(\mathrm{U}(N))}dM = \frac{1}{N!}\Delta^2(u)\prod_{i=1}^{N}du_i\,, \tag{5.24}$$

up to factors of 2 and π. Therefore in (5.23) we have represented the eigenvalue interaction of (5.5) in terms of an infinite number of vertices. Notice however that, at every order in $\hat{g}_s$, only a finite number of vertices contribute, so the perturbation expansion $\sum_{g,h}F_{g,h}\hat{g}_s^{2g-2+h}N^h$ of the partition function can be computed from the hermitean matrix model (5.22) with action (5.23). In order to obtain the perturbation expansion of (5.22), we just bring down the powers of $\mathrm{Tr}\,M^j\,\mathrm{Tr}\,M^k$ from the exponent and we evaluate the vevs with the gaussian weight $\exp(-\frac{1}{2\hat{g}_s}\,\mathrm{Tr}\,M^2)$. The gaussian averages can be computed in many ways, and we review some of these techniques in appendix A.

Let us now consider the expansion around a nontrivial flat connection, focusing on $p=2$ (the general case is similar). The resulting integral is given by (5.4) with $p=2$. Equivalently, we can obtain it by expanding around the critical point $u^* = -i\pi n$ of the exponent in (5.5). We will take the representative of n in the Weyl orbit given by

$$n = (0,\ldots,0,1,\ldots,1) \tag{5.25}$$

where there are N_1 0's and N_2 1's. There are two groups of integration variables, as in (5.4), that we will denote by $\{\lambda_i\}_{i=1,\ldots,N_1}$, and $\{\mu_i\}_{i=1,\ldots,N_2}$. The measure factor in (5.4) reads now:

$$(-1)^{N_1N_2}\prod_{1\le i<j\le N_1}\left(2\sinh\frac{\lambda_i-\lambda_j}{2}\right)^2\prod_{1\le i<j\le N_2}\left(2\sinh\frac{\mu_i-\mu_j}{2}\right)^2\prod_{i,j}\left(2\cosh\frac{\lambda_i-\mu_j}{2}\right)^2. \tag{5.26}$$

The model is then equivalent to a two-matrix model with an $N_1 \times N_1$ hermitean matrix M_1 and an $N_2 \times N_2$ hermitean matrix M_2. The two matrices interact through the last factor in (5.26), that can be written as:

$$\exp\left\{2\sum_{i,j}\log\left(2\cosh\frac{\lambda_i-\mu_j}{2}\right)\right\}. \tag{5.27}$$

In terms of M_1 and M_2, this is

$$W(M_1,M_2)=\sum_{k=1}^{\infty} b_k \sum_{s=0}^{2k}(-1)^s\binom{2k}{s}\,\mathrm{Tr}\,M_1^s\,\mathrm{Tr}\,M_2^{2k-s}\,, \tag{5.28}$$

where

$$b_k=\frac{2^{2k}-1}{k(2k)!}B_{2k}\,. \tag{5.29}$$

On the other hand, M_1 and M_2 interact with themselves through the potentials $V(M_1)$, $V(M_2)$, given in (5.23). Making use of (5.24) we finally obtain an "effective" two-matrix model given by:

$$\begin{aligned}&\frac{1}{\mathrm{vol}(\mathrm{U}(N_1))\times\mathrm{vol}(\mathrm{U}(N_2))}\times \\ &\quad\times\int dM_1 dM_2\exp\left\{-\frac{1}{2\hat{g}_s}\,\mathrm{Tr}\,M_1^2-\frac{1}{2\hat{g}_s}\,\mathrm{Tr}\,M_2^2+V(M_1)+V(M_2)+W(M_1,M_2)\right\}.\end{aligned} \tag{5.30}$$

Similar ideas and techniques to analyze matrix models expanded around nontrivial vacua have been presented in [25] (see also [26]).

In (5.31) we have omitted an overall factor:

$$(-4)^{N_1N_2}\mathrm{e}^{-\frac{\hat{g}_s}{12}N(N^2-1)}\mathrm{e}^{\hat{k}N_2\pi i/2}, \tag{5.31}$$

where the last factor equals $\exp\{\frac{1}{2\hat{g}_s}(u^*)^2\}$, which is the value of the classical CS action on the flat connection associated to (5.25). Notice that the overall factor $\frac{\mathrm{e}^{\hat{k}N_2\pi i/2}}{\mathrm{vol}(\mathrm{U}(N_1))\times\mathrm{vol}(\mathrm{U}(N_2))}$ is in agreement with the prediction for the structure of the semiclassical expansion of CS [16].

Using (5.31), the perturbative expansion around the nontrivial flat connection is just a matter of computing averages in the gaussian ensemble. We have computed the perturbative free energy $F^{\mathrm{pert}}=\sum_g F^{\mathrm{pert}}_{g,h}(N_1,N_2)\hat{g}_s^{2g-2+h}$ up to order 4 in the effective coupling constant. These quantities are homogeneous, symmetric polynomials of degree h in N_1, N_2. For genus 0 one has:

$$\begin{aligned}F^{\mathrm{pert}}_{0,4} &= \frac{1}{288}\left\{N_1^4+6N_1^3N_2+18N_1^2N_2^2+6N_1N_2^3+N_2^4\right\},\\ F^{\mathrm{pert}}_{0,6} &= -\frac{1}{345600}\Big\{4N_1^6+45N_1^5N_2+225N_1^4N_2^2+1500N_1^3N_2^3+\\ &\qquad +225N_1^2N_2^4+45N_1N_2^5+4N_2^6\Big\}.\end{aligned} \tag{5.32}$$

For genus 1, one finds:

$$\begin{aligned} F_{1,2}^{\rm pert} &= -\frac{1}{288}\left\{N_1^2 - 6N_1N_2 + N_2^2\right\}, \\ F_{1,4}^{\rm pert} &= \frac{1}{69120}\left\{2N_1^4 + 105N_1^3N_2 - 90N_1^2N_2^2 + 105N_1N_2^3 + 2N_2^4\right\}. \end{aligned} \tag{5.33}$$

Finally, for genus 2 one finds:

$$F_{2,2}^{\rm pert} = -\frac{1}{57600}\left\{N_1^2 + 60N_1N_2 + N_2^2\right\}. \tag{5.34}$$

As a partial check of these expressions, notice that, if $N_1 = N$ and $N_2 = 0$ (i.e. when we specialize to the trivial connection) the partition function of M_p is identical to the partition function on $\mathbf{S}^3$, up to a rescaling of the coupling constant, and the coefficients $F_{g,h}^{\rm pert}(N)$ can be obtained from the results of [27, 4]. Their explicit expression is

$$\begin{aligned} F_{0,h}^{\rm pert} &= \frac{B_{h-2}}{(h-2)h!} \\ F_{1,h}^{\rm pert} &= -\frac{1}{12}\frac{B_h}{h\,h!} \\ F_{g,h}^{\rm pert} &= -\frac{1}{h!}\frac{B_{2g-2+h}}{2g-2+h}\frac{B_{2g}}{2g(2g-2)}, \qquad g \geq 2, \end{aligned} \tag{5.35}$$

in agreement with the above results for $N_2 = 0$. In the next sections we will see that the above expansions exactly agree with the expansion of the closed string amplitudes on local $\mathbb{P}^1 \times \mathbb{P}^1$ near the orbifold point.

For $\mathbf{S}^3/\mathbb{Z}_p$ with general p, the result for an arbitrary flat connection can be written as a p-matrix model

$$\begin{aligned} &\frac{1}{\prod_{i=1}^p \mathrm{vol}(\mathrm{U}(N_i))}\times \\ &\times \int \prod_{i=1}^p dM_i \exp\left\{-\frac{1}{2\hat{g}_s}\sum_{i=1}^p \mathrm{Tr}\, M_i^2 + \sum_{i=1}^p V(M_i) + \sum_{1\leq i<j\leq p} W(M_i, M_j)\right\}, \end{aligned} \tag{5.36}$$

where $V(M)$ is still given by (5.23), and $W(M_i, M_j)$ is given by

$$W(M_i, M_j) = \sum_{k=1}^\infty 2^{-k+1} a_k^{(ij)} \sum_{s=0}^k (-1)^s \binom{k}{s} \mathrm{Tr}\, M_1^s\, \mathrm{Tr}\, M_2^{k-s}, \tag{5.37}$$

and $a_k^{(ij)}$ are the coefficients in the Taylor series expansion of

$$\log\sinh\left((j-i)\frac{\pi i}{p} + x\right). \tag{5.38}$$

6. Closed topological strings on $\mathcal{O}(-K) \to \mathbb{P}^1 \times \mathbb{P}^1$.

In this section we will calculate the topological string amplitudes for the non-compact Calabi-Yau geometry which is the large-N dual of $T^*\mathbf{S}^3/\mathbb{Z}_2$, by using mirror symmetry and the B-model technique. The geometry is the canonical line bundle over $F_0 = \mathbb{P}^1 \times \mathbb{P}^1$. The B-model mirror description of that geometry is encoded in a Riemann surface with a meromorphic differential. Many of the techniques developed here extend to more general non-compact Calabi-Yau geometries.

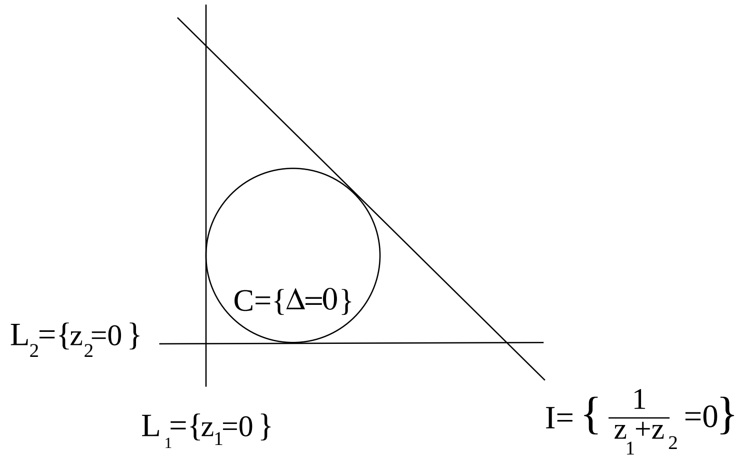


Figure 4: Schematic view of the unresolved moduli space of $\mathcal{O}(-K) \to \mathbb{P}^1 \times \mathbb{P}^1$.

6.1 Moduli space of $\mathcal{O}(-K) \to F_0$

Let us first describe the complexified Kähler moduli space.

The method to analyze the complexified Kähler moduli space is to study the complex structure deformations of the mirror as encoded in the period integrals. Up to finite choice of integration constants these periods are captured by the linear differential operators of order two [28][7]

$$\begin{aligned}\mathcal{L}_1 &= z_2(1-4z_2)\xi_2^2 - 4z_1^2\xi_1^2 - 8z_1z_2\xi_1\xi_2 - (6z_1+6z_2)\xi_1 + \xi_2\,,\\ \mathcal{L}_2 &= z_1(1-4z_1)\xi_1^2 - 4z_2^2\xi_2^2 - 8z_1z_2\xi_1\xi_2 - (6z_1+6z_2)\xi_2 + \xi_1\,,\end{aligned} \tag{6.1}$$

where the $\xi_i = \partial/\partial z_i$. Differential systems governing the periods can have only regular singular points [30], i.e. the periods will in "suitable" coordinates have at worst (in this case double) logarithmic singularities. One can obtain the corresponding singular locus by calculating the resultant of the leading (order two) pieces of $\mathcal{L}_i = 0$ with ξ_i viewed as algebraic variables. This yields

$$z_1z_2\left[1-8(z_1+z_2)+16(z_1-z_2)^2\right] =: z_1z_2\Delta = 0\,.$$

We need to compactify the z_1, z_2 space and chose $\mathbb{P}^2$ as first approximation to do that, i.e. we consider in addition the patches $(a_1 = 1/z_2, a_2 = z_1/z_2)$ and $(b_1 = 1/z_1, b_2 = z_2/z_1)$. Transforming (6.1) and repeating the analysis in these coordinates we get the following schematic picture of the degeneration locus in figure 4. We see that the C touches L_1 at $z_2 = 1/4$, L_2 at $z_1 = 1/4$ and I at $u = \frac{z_1}{z_1+z_2} = 1/2$. All intersections are with contact order two. For example identifying[8] at $C \cap I$ $a = 4(1-2u)$ and $b = \frac{8}{z_1+z_2}$ the local equations at the intersection $C \cap I$ are

$$C = \{a^2 - b = 0\} \quad \text{and} \qquad I = \{b = 0\}\,. \tag{6.2}$$

[7]The Picard-Fuchs equations as starting point of the further considerations can be easily obtained for all toric non-compact Calabi-Yau spaces [29]. Using the mirror geometry given most explicitly in [18], it would be also possible to work directly with period integrals.

[8]Similarly at $C \cap L_2$ we set $a = (1-4z_1)$ and $b = z_2$ and at $C \cap L_1$, $a = (1-4z_2)$ and $b = z_1$.

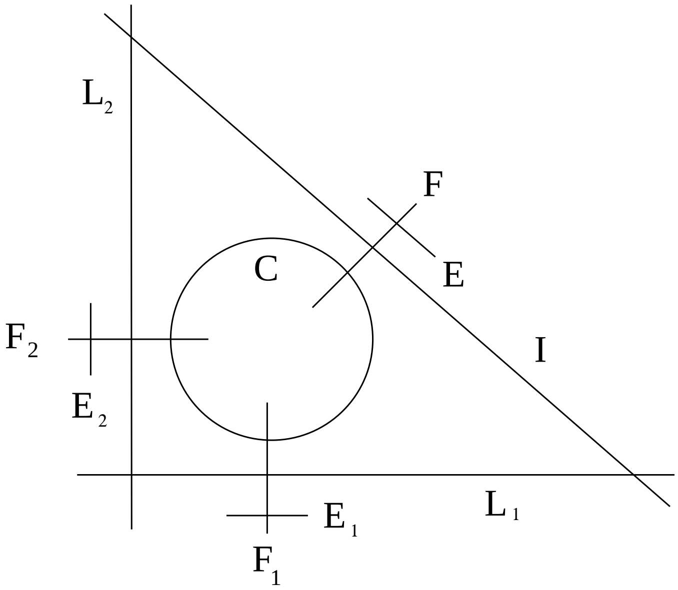


Figure 5: Schematic view of the resolved moduli space of $\mathcal{O}(-K) \to \mathbb{P}^1 \times \mathbb{P}^1$.

As a consequence the differential equations are not solvable in the local variables (a, b). Physically speaking we have to consider a multi scaling limit in approaching the intersection point in order to be able to define the $F^{(g)}$.

In algebraic geometry this corresponds to the well-known fact that one can resolve the moduli space of Kuranishi family in a way that all boundary divisors, i.e. the discriminant components, have normal crossings. The vanishing coordinates at those divisors are the "suitable" coordinates for the statement about the regular singular behavior of the periods above. The resolution process of (6.2) is standard and was used in similar context in [31]. To resolve points of contact order k one introduces k times $(a_i : b_i)$ homogeneous P^1 variables and k relations. In our case the process produces normal crossing after introducing $a_1 a = b_1 b$ and $a_2 a = b_2 a_1$. In the $(a, b|a_1 : b_1|a_2 : b_2)$ variables the coordinates along the divisors are C: $(\sqrt{b}, b|a : 1, |1 : 1)$, I: $(a, 0|0 : 1|0 : 1)$, E: $(0, 0|a_1 : 1|1 : 0)$ and finally F: $(0, 0|0 : 1|a_2 : b_2)$. One sees from that that F intersects I, E and C in the way depicted in figure 5. The blow ups of $L_i \cap C$ are completely analogous resolutions of the local equation (6.2).

For us the most relevant points are

$$
\begin{array}{rll}
I \cap F: & \mathbb{Z}_2 \,\text{orbifold pt.}: & \text{Matrix model expansion,} \\
L_1 \cap L_2: & \text{large complex st. pt.}: & \text{Topological A-model expansion.}
\end{array}
$$

Also interesting are the two copies of Seiberg-Witten field theory embedding

$$
\begin{array}{rll}
L_i \cap F_i: & \text{SW weak coupl. pt.}: & \text{Space time instanton expansion,} \\
L_i \cap C: & \text{SW strong coupl. pt.}: & \text{SW strong coupling expansion scheme.}
\end{array}
$$

6.2 Choosing local complex structure coordinates

Choosing local complex structure coordinates is merely a technical issue needed to evaluate the periods at all points, two of which will really become the good physical B-model variables. The transversal directions to the divisors are the good complex coordinates coordinates. At $I \cap F$, $b_2 = 1$ and a_2 moves transversally to I along F so $a_2 = a_1/a = b/a^2$ and a moves transversally to F along I, i.e. $(b/a^2, a)$ are good coordinates. At $F \cap E$ $b_2 = a^2/b$ is transverse to E and $a_1 = b/a$ transverse to F, good coordinates are $(a^2/b, b/a)$ and finally at $C \cap F$. And at $C \cap I$: $(1 - b/a^2, a)$ are good coordinates. This clarifies the choice of the complex structure variables at all blown loci. At $L_1 \cap L_2$ good local variables are (z_1, z_2) and at $L_i \cap I$ $\left(\frac{z_i}{z_1+z_2}, \frac{1}{z_1+z_2}\right)$. Clearly the right choice of these variables is a local issue, e.g. we could also have chosen $\left(\frac{z_1}{z_2}, \frac{1}{z_2}\right)$ at $L_1 \cap I$ which differs only away from L_1 from the previous ones.

A global issue in the choice of complex parameters is the fact that (z_1, z_2) are actually $\mathbb{Z}_2 \times \mathbb{Z}_2$ multi covering variables. The branching loci of which give rise to the E-type divisors. Choosing single cover variables $x_1^2 = z_1$ and $x_2^2 = z_2$ the conifold locus $\Delta = 0$ reduces into four components and the embedding of the Seiberg-Witten u-planes (F_1, F_2 in figure 5.) become more familiar since there is now a $(1, -1)$ dyon component and a $(0, -1)$ monopole component crossing the four u-planes in the single cover variables.

6.3 Solving the Picard-Fuchs equation near the orbifold point

In the usual application of mirror symmetry the periods are evaluated near the large complex structure point $L_1 \cap L_2$. Two of the periods, usually called $t_1 = \log(z_1) + \mathcal{O}(z)$ and $t_2 = \log(z_2) + \mathcal{O}(z)$, approximate at this point the classical large Kähler volumes of the two $\mathbb{P}^1$.

Here we need to expand the solutions to the Picard-Fuchs equations near the orbifold point. It is convenient to use the variables

$$\left(x_1 = 1 - \frac{z_1}{z_2},\, x_2 = \frac{1}{\sqrt{z_2}(1 - z_1/z_2)}\right). \tag{6.3}$$

The choice of x_1 and (B.5) ensures that $q_1 = q_2$ or $t_1 = t_2$ near the expansion point, while the vanishing of x_2 ensures that $\sqrt{z_2}$ goes faster to infinity then x_1 goes to zero.

The periods in this variables have the following structure

$$\begin{aligned}
\omega_0 &= 1\,, \\
s_1 &= -\log(1 - x_1) = \sum_m c_{m,0} x_1^m = t_1 - t_2\,, \\
s_2 &= \sum_{m,n} c_{m,n} x_1^m x_2^m\,, \\
F^{(0)}_{s_2} &= s_2 \log(x_1) + \sum_{m,n} d_{m,n} x_1^m x_2^n\,,
\end{aligned}$$

where the $c_{m,n}$ and $d_{m,n}$ are determined by the following recursions relations

$$c_{m,n} = c_{m-1,n} \frac{(n + 2 - 2m)^2}{4(m - n)(m - 1)}\,,$$

$$
\begin{aligned}
c_{m,n} &= \frac{1}{n(n-1)}(c_{m,n-2}(n-m-1)(n-m-2) - c_{m-1,n-2}(n-m-1)^2)\,, \\
d_{m,n} &= \frac{d_{m-1,n}(n+2-2m)^2 + 4(n+1-2m)c_{m,n} + 4(2m-n-2)c_{m-1,n}}{4(m-n)(m-1)}\,, \\
d_{m,n} &= \frac{1}{n(n-1)}\Big(d_{m,n-2}(n-m-1)(n-m-2) - d_{m-1,n-2}(n-m-1)^2 + \\
&\qquad + (2n-2-2m)c_{m-1,n-2} + (2m+3-2n)c_{m,n-2}\Big)\,.
\end{aligned}
$$

Up to linear transformations we expect the s_1 and s_2 periods to be the good coordinates in which we will express the B-model correlators, which are giving in (x_1, x_2) coordinates using (B.3) and (6.3). We therefore need the inverse function $x(s)$. To invert the second and third period we define $\widetilde{s}_1 = s_1 = x_1 + \mathcal{O}(x^2)$ and $\widetilde{s}_2 = s_1/s_2 = x_2 + \mathcal{O}(x^2)$

$$
\begin{aligned}
x_1(s_1) &= 1 - e^{-\widetilde{s}_1}\,, \\
x_2(\widetilde{s}_1, \widetilde{s}_2) &= \widetilde{s}_2 + \frac{1}{4}\widetilde{s}_1\widetilde{s}_2 + \frac{1}{192}\widetilde{s}_1^2\widetilde{s}_2 - \frac{1}{256}\widetilde{s}_1^3\widetilde{s}_2 - \frac{49}{737280}\widetilde{s}_1^4\widetilde{s}_2 - \frac{1}{192}\widetilde{s}_1^2\widetilde{s}_2^3 + \mathcal{O}(\widetilde{s}^6)\,.
\end{aligned} \tag{6.4}
$$

This yields the mirror map at the orbifold point.

6.4 The genus zero partition function at the orbifold point

The genus zero partition function can now be obtained by transforming (B.3) using (6.3) to the (x_1, x_2) and by (6.4) to the (s_1, s_2) coordinates. These s variables are flat coordinates, which have natural $\mathrm{GL}(2, \mathbb{C})$ structure. It follows that we can integrate the $c_{ijk}(s) = \partial_{s_i}\partial_{s_j}\partial_{s_k}F^{(0)}$ to obtain the prepotential $F^{(0)}$ up to a quadratic polynomial in s. The appropriate variables S_1, S_2 that match the 't Hooft parameters in the CS/matrix model side are given by

$$
S_1 = \frac{1}{4}(s_1 + s_2)\,, \qquad S_2 = \frac{1}{4}(s_1 - s_2)\,. \tag{6.5}
$$

In view of these identifications, the fact that $s_1 = t_1 - t_2$ and the symmetry of S_1, S_2 in the partition functions below we conclude that $s_2 = t_1 + t_2$, hence $S_i = 1/2t_i$. This can be shown also by analytic continuation.

An alternative way to get $F^{(0)}$ is to integrate $F^{(0)}_{s_2}$ with respect to the flat coordinate s_2. This way one misses terms, which depend only on s_1, but those can be reinstalled by requiring symmetry between S_1 and S_2 in the final expression. So one can get $F^{(0)}$ up to a constant. By comparing the all genus partition function $F = \sum_{i=0}^{\infty} g_s^{2g-2}F^{(g)}$ with the matrix model one also has to make a choice of the string coupling g_s namely $g_s^{\mathrm{top}} = 2i\hat{g}_s$. This way the terms in front of $\hat{g}_s^{-2}$ are

$$
F^{(0)} = \frac{1}{2}(S_1^2\log(S_1) + S_2^2\log(S_2)) + \sum_{m,n} c^{(0)} S_1^m S_2^n + p_2(S)\,. \tag{6.6}
$$

The $c^{(0)}_{m,n}$ are only non-zero for $n + m \in 2\mathbb{Z}$ and symmetric in m, n. The first few degrees have been checked against the matrix model calculation in (5.32):

$$
d = 4\,:\, \frac{1}{288}(S_1^4 + 6S_1^3S_2 + 18S_1^2S_2^2 + 6S_1S_2^3 + S_2^4)
$$

$$d=6: -\frac{1}{345600}(4S_1^6+45S_1^5S_2+225S_1^4S_2^2+1500S_1^3S_2^3+\cdots)$$
$$d=8: \frac{1}{40642560}(4S_1^8+63S_1^7S_2+441S_2^6S_1^2+441S_1^5S_2^3+30870S_1^4S_2^4+\cdots)\,.$$

Both calculations are in perfect agreement.

6.5 The genus one B-model amplitude

According to [32, 12] and taking the simplification in the local case [33] into account we expect the holomorphic $\bar{s}_i \to 0$ limit of the topological amplitude to be

$$F^{(1)} = \log\left(\det^{\frac{1}{2}}\left(\frac{\partial x_i}{\partial s_j}\right)\Delta(x_1,x_2)^{-\frac{1}{12}}\prod_{i=1}^{2} x_i^{b_i}\right)$$

where the conifold discriminant is given by $\Delta = (16 - 16x_2^2 + 8x_1x_2^2 + x_1^2x_2^4)$ in the x coordinates. The exponent $-1/12$ at the conifold is universal and $b_1 = 1/3$, $b_2 = 0$. Note that the rescaling of the string coupling does not affect this comparison with the expression from the matrix model. Expanding in the matrix model flat coordinates (S_1, S_2) and get

$$F^{(1)} = -\frac{1}{12}(\log S_1 + \log S_2) + \sum_{m,n} c^{(1)}_{m,n} S_1^m S_2^n\,.$$

Again the $c^{(1)}_{m,n}$ are only non-zero for $n+m \in 2\mathbb{Z}$ and symmetric in m,n. The first few degrees are given by

$$d=2: -\frac{1}{288}(S_1^2-6S_1S_2+S_2^2)$$
$$d=4: \frac{1}{69120}(2S_1^4+105S_1^3S_2-90S_1^2S_2^2+105S_1S_2^3+2S_2^4)$$
$$d=6: -\frac{1}{17418240}(8S_1^6-189S_1^5S_2+7560S_1^4S_2^2-630S_1^3S_2^3+\cdots)$$
$$d=8: \frac{1}{1857945600}(16S_1^8+435S_1^7S_2-27195S_2^6S_1^2+196770S_1^5S_2^3+222600S_1^4S_2^4+\cdots)\,.$$

in perfect agreement with the matrix model calculation (5.33).

6.6 The higher genus topological B-model amplitudes at the orbifold point

The key problem in deriving higher genus results in the B-model with multi dimensional moduli space is to find the propagators of the topological B-model. Due to the technical nature of the problem we relegate the derivation of the propagators in the appendix B.

Equipped with $F^{(0)}$, $F^{(1)}$ and the propagator (B.7) $S := S^{22}$ we can readily calculate $F^{(2)}$. Since we assured the same singular behavior of the propagator, the ambiguity at genus 2 has not to be determined again, but can be taken after suitable coordinate transformation from the calculation of the $F^{(2)}$ at the large complex structure.

$$\begin{aligned} F^{(2)} &= -\frac{1}{8}S_2^2F^{(0)}_{,4} + \frac{1}{2}S_2F^{(1)}_{,2} + \frac{5}{24}S_2^3(F^{(0)}_{,3})^2 - \frac{1}{2}S_2^2F^{(1)}_{,1}F^{(0)}_{,3} + \frac{1}{2}S_2(F^{(1)}_{,1})^2 + f^{(2)} \\ &= -\frac{1}{240}\left(\frac{1}{S_1^2}+\frac{1}{S_2^2}\right) + \sum_{m,n} c^{(2)}_{m,n} S_1^m S_2^n \end{aligned}$$

The $c^{(2)}_{m,n}$ are only non-zero for $n+m\in 2\mathbb{Z}$ and symmetric in m,n

$$
\begin{aligned}
&d=2:\ -\frac{1}{57600}(S_1^2+60S_1S_2+S_2^2)\\
&d=4:\ \frac{1}{1451520}(S_1^4+126S_1^3S_2+378S_1^2S_2^2+126S_1S_2^3+S_2^4)\\
&d=6:\ -\frac{1}{2654208000}(64S_1^6-38385S_1^5S_2+334575S_1^4S_2^2+124500S_1^3S_2^3+\cdots)\\
&d=8:\frac{1}{81749606400}(64S_1^8+68343S_1^7S_2-2224299S_2^6S_1^2+7547001S_1^5S_2^3+27188870S_1^4S_2^4+\cdots).
\end{aligned}
$$

The $d=2$ term and the terms involving only one S_i are again in perfect agreement with the matrix model (5.34)(5.35).

The iteration in the genus is in principle no problem in the B-model, however one has to fix the holomorphic ambiguity at each genus, which we pushed only up to genus 3.

$$
\begin{aligned}
F^{(3)} =\ & S_2F^{(2)}_{,1}F^{(1)}_{,1}-\frac{1}{2}S_2^2F^{(2)}_{,1}F^{(0)}_{,3}+\frac{1}{2}S_2F^{(2)}_{,2}+\frac{1}{6}S_2^3(F^{(1)}_{,1})^3F^{(0)}_{,3}-\frac{1}{2}S_2^2F^{(1)}_{,2}(F^{(1)}_{,1})^2-\\
&-\frac{1}{2}S_2^4(F^{(1)}_{,1})^2(F^{(0)}_{,3})^2+\frac{1}{4}S_2^3(F^{(1)}_{,1})^2F^{(0)}_{,4}+S_2^3F^{(1)}_{,2}F^{(1)}_{,1}F^{(0)}_{,3}-\frac{1}{2}S_2^2F^{(1)}_{,3}F^{(1)}_{,1}-\\
&-\frac{1}{4}S_2^2(F^{(1)}_{,2})^2+\frac{5}{8}S_2^5F^{(1)}_{,1}(F^{(0)}_{,3})^3-\frac{2}{3}S_2^4F^{(1)}_{,1}F^{(0)}_{,4}F^{(0)}_{,3}-\frac{5}{8}S_2^4F^{(1)}_{,2}(F^{(0)}_{,3})^2+\\
&+\frac{1}{4}S_2^3F^{(1)}_{,2}F^{(0)}_{,4}+\frac{5}{12}S_2^3F^{(1)}_{,3}F^{(0)}_{,3}+\frac{1}{8}S_2^3F^{(0)}_{,5}F^{(1)}_{,1}-\frac{1}{8}S_2^2F^{(1)}_{,4}-\frac{7}{48}S_2^4F^{(0)}_{,5}F^{(0)}_{,3}+\\
&+\frac{25}{48}S_2^5F^{(0)}_{,4}(F^{(0)}_{,3})^2-\frac{5}{16}S_2^6(F^{(0)}_{,3})^4-\frac{1}{12}S_2^4(F^{(0)}_{,4})^2+\frac{1}{48}S_2^3F^{(0)}_{,6}+f^{(3)}\\
=\ &-\frac{1}{1008}\left(\frac{1}{S_1^2}+\frac{1}{S_2^2}\right)+\sum_{m,n}c^{(2)}_{m,n}S_1^mS_2^n\ .
\end{aligned}
$$

The first few coefficients are

$$
\begin{aligned}
&d=4:\ \frac{1}{557383680}(16S_1^4-345S_1^3S_2+58500S_1^2S_2^2-345S_1S_2^3+16S_2^4)\\
&d=6:\ -\frac{1}{36787322880}(64S_1^6-325116S_1^5S_2+1461735S_1^4S_2^2-2198130S_1^3S_2^3+\cdots)\,.
\end{aligned}
$$

These results are predictions for the matrix model.

7. Some generalizations

7.1 Adding matter

The considerations in the preceding sections can be easily generalized by adding matter fields. In terms of Chern-Simons theory this has been discussed in [8, 9]. In this section we consider this in the mirror B-model language. We will show that all the amplitudes computed in [8, 9] are matrix model amplitudes. This includes invariants for torus knots and links in the classes of three- manifolds M considered in this paper.

For definiteness, consider the B-model geometry corresponding to

$$xz=(e^u-1)(e^{u-t_1}-1)(e^v-1)(e^{v-t_2}-1)\,, \tag{7.1}$$

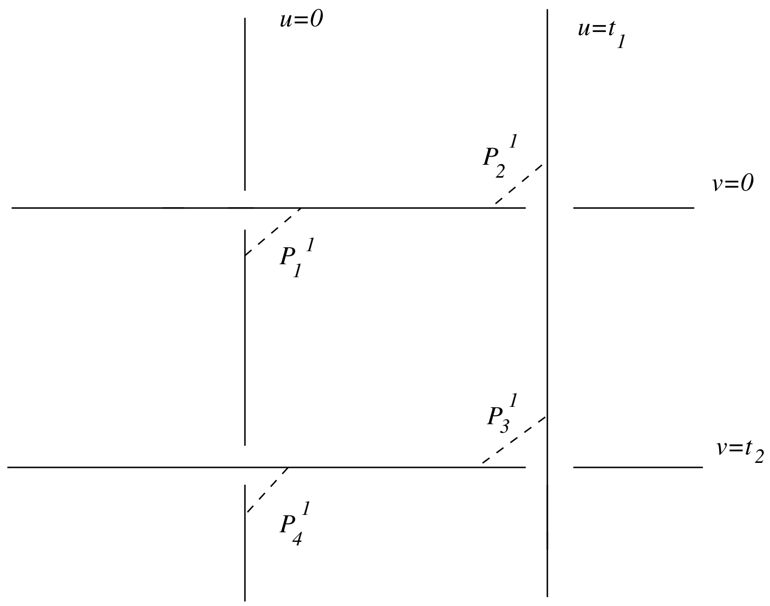


Figure 6: The figure depicts the four isolated $\mathbb{P}^1$'s in the Calabi-Yau (7.1).

which is a mirror of the A-model geometry studied in section 7.5 of [8]. There are four holomorphic $\mathbb{P}^1$'s corresponding to four points with $u = 0, t_1$ and $v = 0, t_2$. We can consider wrapping some numbers N_i D-branes on the i-th $\mathbb{P}^1$.

Consider the partition function of the modes corresponding to the $\mathbb{P}^1$ at $u = 0 = v$. This is given by

$$Z(S) = \langle 0_v|S|0_v\rangle = \langle 0_u|0_v\rangle\,,$$

since u and v are related by S operator given in (2.5). Alternatively, the wave function $|0_u\rangle$ is obtained from $|0_v\rangle$ by simply exchanging u and v, as this is what S does, up a constant. We have that

$$Z = \frac{1}{|\mathcal{W}|}\int\prod_i \frac{du_i dv_i}{(2\pi g_s)^{\frac{1}{2}}}\Delta_H(u)\Delta_H(v)\, e^{\sum_i u_i v_i/g_s}, \tag{7.2}$$

where we used that u and v are canonically conjugate, so $\langle u|v\rangle = e^{\sum_i u_i v_i/g_s}$, and furthermore $|\mathcal{W}| = N!$ is the order of the Weyl group of $\mathrm{U}(N)$ where $N = N_1$ is the number of wrapped D-branes. It is easy to see that the partition functions of the modes living on the other $\mathbb{P}^1$'s in (7.1) coincide with (7.2) with appropriate values of N.

This corresponds to a matrix model given by

$$Z = \frac{1}{\mathrm{vol}(\mathrm{U}(N))}\int_{[u,v]=0}\frac{\hat{d}u\hat{d}v}{(2\pi g_s)^{\frac{N}{2}}}\,, e^{\mathrm{Tr}\,uv/g_s}, \tag{7.3}$$

where the integral is over commuting hermitean matrices u and v. The measure in the path integral is defined as follows. Consider the space of unitary matrices U, V. where $U = e^u$ and $V = e^v$. Since u, v are canonically conjugate it is natural to consider the symplectic form

$$\omega = \mathrm{Tr}\, U^{-1}dU \wedge V^{-1}dV\,, \tag{7.4}$$

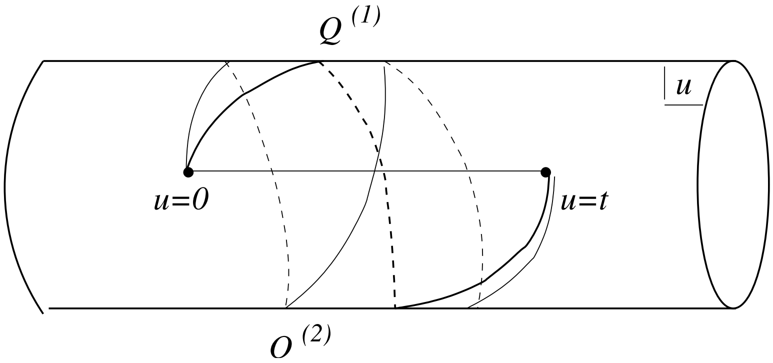


Figure 7: The figure depicts the lift in the full geometry of the line passing through the north and the south pole of the $\mathbb{P}^1_1$ and $\mathbb{P}^1_2$. This is the u-cylinder intersected at $u = 0$ and u_1 by the two $\mathbb{P}^1$'s. Consequently there is a family of BPS strings connecting the two $\mathbb{P}^1$'s and winding around the cylinder. The strings are labeled by their winding number.

in terms of the left $\mathrm{U}(N)$ invariant line elements $U^{-1}dU$ and $V^{-1}dV$. The symplectic form gives rise to the volume element on the phase space $\omega^d/d!$ where $d = N^2$ is the dimension of $\mathrm{U}(N)$. The measure in the path integral (7.3) is induced from this by restricting to the space of commuting matrices U and V. Namely, u and v commute, there exists a unitary matrix Ω that diagonalizes both U and V, i.e. $\Omega U \Omega^{-1} = \mathrm{diag}(u_i)$ and $\Omega V \Omega^{-1} = \mathrm{diag}(v_i)$. The volume of the phase space is obtained by writing (7.4) in terms of Ω, u_i and v_i. Integrating over Ω to reduce the path integral to integral over the eigenvalues recovers (7.2). This is akin to the matrix models studied in [34] based on hermitean matrices. In the following, the measure on the phase space of pairs of conjugate, $\mathrm{U}(N)$ Lie-algebra valued variables, u, v will be denoted by $\hat{du}\hat{dv}$. In particular, it should be understood that u and v commute. The fact that u and v are commuting matrices in (7.3) is natural as the $A_{\bar{z}}$ equation of motion implies that the matrix $[u(z), v(z)]_{ij}$ vanishes,[9] and we have localized to zero modes.

Because there is more than one stack of B-branes, there are additional open string sectors with the two ends of the string on the D-branes wrapping the different $\mathbb{P}^1$'s. By the same arguments as in [8], the only modes that contribute to the B-model amplitudes correspond to the strings stretching between $\mathbb{P}^1_i$ and $\mathbb{P}^1_{i+1}$ in the figure 6.

Consider for example the strings stretching between $u = 0$ and $u = t_1$ on the u-cylinder and connecting $\mathbb{P}^1_1$ and $\mathbb{P}^1_2$. There are different topological sectors of these strings — corresponding to how many times the string winds around the $\mathbf{S}^1$, see figure 7. From each sector we get one physical scalar in the bifundamental representation. Moreover, each of the strings is minimally coupled to the gauge-fields on the spheres it ends on. Thus, the matter part of the action is

$$S(Q_{12}, u) = \sum_n \mathrm{Tr}\, Q_{12}^{(n)} \left((2\pi i n + t)\, 1_1 \otimes 1_2 + \, u_1 \otimes 1_2 - \, u_1 \otimes 1_2\right) Q_{21}^{(n)} \,,$$

where $[Q_{12}^{(n)}]^\dagger = Q_{21}^{(n)}$, and u_1 and u_2 are matrices corresponding to the positions of the first and the second stack of D-branes on the u-cylinder. Recall that there is a relative shift by

[9]This is mirror to the vanishing of F in (2.4).

parameter t between them that contributes to the mass. The contribution of this to the path integral $\mathcal{O}_{12} = \int \prod_n dQ_{12}^{(n)} \exp(S(Q_{12}, u)/g_s)$ is trivial to evaluate, as in [35], giving

$$\begin{aligned} \mathcal{O}_{12} &= \exp\left\{\sum_{n=1}^{\infty} \frac{\mathrm{e}^{-nt}}{n} \operatorname{Tr} U_1^n \operatorname{Tr} U_2^{-n}\right\} \\ &= \sum_R e^{-l_R t} \operatorname{Tr}_R U_1 \operatorname{Tr}_R U_2 \,, \end{aligned} \tag{7.5}$$

where $U_{1,2} = e^{u_{1,2}}$ and l_R is the number of boxes in representation R. In writing (7.5) we used the regularization $\sum_n \log(2\pi i n + x) = \log\sinh(x) + \mathit{const}$ of the one loop path integral.

Note that mirror symmetry transforms the tower of modes above to a single string ground state propagating on a circle, where as above $u_{1,2}$ get related to Wilson lines on the mirror $\mathbf{S}^3$'s as in [8]: $S(Q_{12}, A) = \oint_\gamma \operatorname{Tr} Q_{12} \Big((d+t)1_1 \otimes 1_2 - A_1 \otimes 1_2 + 1_1 \otimes A_2\Big) Q_{21}$. This is the expected action of T-duality on D-branes. The operators $U_{1,2}$ are now interpreted as Wilson loop operators in Chern-Simons theory.

The matrix model allows one to very simply calculate expectation values of Wilson loop operators. Consider for example evaluating the Chern-Simons path integral on $\mathbf{S}^3$ in the presence of a Hopf link. This can be obtained from gluing two solid 2-tori by an S transformation, and in the presence of Wilson loops in representations R and R' on the one-cycles that cannot shrink, i.e. we are interested in evaluating $\langle R|S|R'\rangle = \langle R_v \mid R'_u\rangle$. In the light of the discussion above, the path integral on the solid two-torus with the Wilson loop is mirror to computing

$$|R_v\rangle = \frac{1}{|\mathcal{W}|^{\frac{1}{2}}} \int \prod_i du_i \operatorname{Tr}_R U \Delta_H(u) |u\rangle \,,$$

so that $\langle R|S|R'\rangle$ is

$$\langle R|S|R'\rangle = \frac{1}{\mathrm{vol}(\mathrm{U}(N))} \int \frac{\hat{d}u\hat{d}v}{(2\pi g_s)^{\frac{N}{2}}} \operatorname{Tr}_R U e^{\operatorname{Tr} uv/g_s} \operatorname{Tr}_{R'} V^{-1} \,. \tag{7.6}$$

It is easy to see that this agrees with the expression (2.12) – as above it can be evaluated simply by using the Weyl character formula, for $\mathrm{U}(N)$

$$\operatorname{Tr}_R U = \frac{\sum_{w\in\mathcal{W}} \epsilon(w) \mathrm{e}^{iw(\alpha)\cdot u}}{\sum_{w\in\mathcal{W}} \epsilon(w) \mathrm{e}^{iw(\rho)\cdot u}} \,,$$

where α is the highest weight vector of the representation R of $\mathrm{U}(N)$ shifted by ρ. As a check, note that

$$\begin{aligned} \langle R|S|R'\rangle &= \frac{1}{|W|} \left(\frac{g_s}{2\pi}\right)^{\frac{N}{2}} \sum_{w,w'\in\mathcal{W}} \epsilon(ww') \mathrm{e}^{-g_s w(\alpha)\cdot w'(\beta)} \\ &= \left(\frac{g_s}{2\pi}\right)^{\frac{N}{2}} \sum_{w\in\mathcal{W}} \epsilon(w) \mathrm{e}^{-g_s w(\alpha)\cdot\beta}. \end{aligned} \tag{7.7}$$

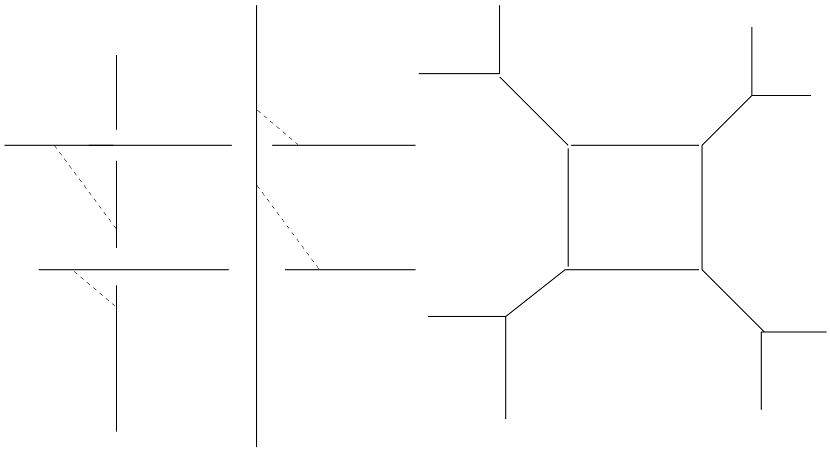


Figure 8: The figure depicts the geometric transition of the open string geometry in figure 6. The geometric transition is a large-N duality, and the matrix model computes amplitudes of the A-model version of geometry on the left (a toric $\mathbb{B}_5$) to all genera.

Note that this exactly agrees with (2.12).

To summarize, we have a matrix model expression for the B-branes in the geometry (7.1), given by

$$\begin{aligned} Z_{\text{tot}} &= \frac{1}{\prod_{i=1}^{4}\text{vol}(\text{U}(N_i))}\sum_{R_1,\ldots,R_4}\int_{[u_i,v_i]=0}\prod_{i=1}^{4}\frac{\hat{d}u_i\hat{d}v_i}{(2\pi g_s)^{\frac{N_i}{2}}}\; e^{-\sum_{i=1}^{4}\text{Tr}\,u_i v_i/g_s}\times \\ &\quad\times e^{-l_1 t_1}\,\text{Tr}_{R_1}U_2^{-1}\,\text{Tr}_{R_2}V_2 e^{-l_2 t_2}\,\text{Tr}_{R_2}V_3^{-1}\,\text{Tr}_{R_3}U_3\times \\ &\quad\times e^{-l_3 t_1}\,\text{Tr}_{R_3}U_4^{-1}\,\text{Tr}_{R_4}V_4\, e^{-l_4 t_2}\,\text{Tr}_{R_4}V_1^{-1}\,\text{Tr}_{R_1}U_1 \end{aligned} \tag{7.8}$$

where the expectation values of the Hopf link operators are computed by matrix integrals (7.7). The minus sign in the exponent corresponds to the fact that the gluing operator is S^{-1} [8].

Moreover, the large-N dual of this is a *mirror* of toric Calabi-Yau geometry without any D-branes, corresponding to a $O(-K)\to\mathbb{P}^1\times\mathbb{P}^1$ blown up at four points, which is a non-generic Del-Pezzo surface $\mathbb{B}_5$, see figure 8. In [8] it was shown that Chern-Simons theory on the A-model mirror of geometries in (7.1) computes the topological A-model amplitudes in $\mathbb{B}_5$ to all genera. What we have shown here is that all genus $\mathbb{B}_5$ amplitudes are really computed by a matrix model!

The above Hopf-link computation is also easy to generalize to more general (m,n) torus knots, where the corresponding operator is

$$|R;m,n\rangle = \text{Tr}_R\, e^{mu+nv}|0_v\rangle\,.$$

where we have picked one particular ordering of operators. Different orderings of the operators differ from this by overall phases.

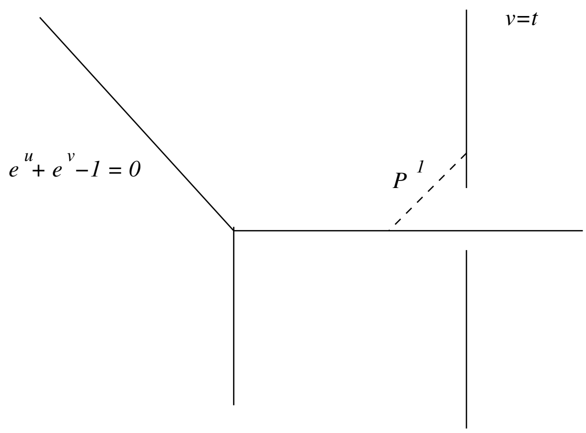


Figure 9: An example of more general geometries.

7.2 More general geometries

The considerations above can be generalized to arbitrary backgrounds of the form

$$xz = \prod_i P_i(u,v)\,.$$

We have a collection of $\mathbb{P}^1$'s where curves

$$P_i = 0 = P_j$$

intersection In general there are also matter multiplets corresponding to strings stretching between the $\mathbb{P}^1$'s whose poles lie on the same curve, and we get a quiver theory. On the nodes of the quiver we get a matrix model,

$$Z_{ij} = \langle 0_{P_i} \mid 0_{P_j} \rangle\,.$$

For example, if $P_i = 0$ and $P_j = 0$ are given respectively by

$$v = W_i'(u)\,, \qquad u = W_j'(v)\,, \tag{7.9}$$

then

$$\begin{aligned} Z_{ij} &= \langle 0_v \mid \mathrm{e}^{W_i(u)/g_s}\, \mathrm{e}^{W_j(v)/g_s} \mid 0_u \rangle \\ &= \frac{1}{\mathrm{vol}(\mathrm{U}(N))} \int \frac{\hat{d}u\, \hat{d}v}{(2\pi g_s)^{\frac{N}{2}}} \exp\left\{ \frac{1}{g_s} \left(\mathrm{Tr}\, W_i(u) - \mathrm{Tr}\, uv + \mathrm{Tr}\, W_j(v) \right) \right\}, \end{aligned} \tag{7.10}$$

This corresponds to replacing T^*M by more general geometries which approximate this in the immediate neighborhood of M.

As an example, let us consider the B-model geometry studied previously in [10], corresponding to a blowup of

$$xz = (e^u + e^v - 1)(e^{v-t} - 1)\,,$$

we have $W_1(v) = -\sum_n \frac{1}{n^2} e^{nv}$ and $W_2(u) = tu$. Note that the gluing operator is related to the superpotential of the theory. Namely, the superpotential for a B-brane in this geometry was computed in [10], where it was found that

$$W(u) = \int_B \Omega = \int (v_2(u) - v_1(u)) du \,,$$

where

$$v_1 = \log(1 - e^u)\,, \qquad v_2 = t\,,$$

are the equations of the corresponding Riemann surfaces. In calculating the superpotential, we kept v fixed over the whole $\mathbb{P}^1$. Note that on the Riemman surface, if we put $v_1(u) = \frac{d}{du} W_1^D(u)$, we have that $W_1^D(u)$ and $W_1(v)$ are Legendre transforms

$$W_1^D(u) = uv + W_1(v)\,.$$

7.3 Framing dependence

In the formalism we have been developing there is a subtlety related to framing dependence. Note that there is more than one operator having the property that it conjugates v to $pu + qv$. For example, given an operator $U_{(p,q)}$ that conjugates v to $pu + qv$, operator $U'_{(p,q)}$

$$U'_{(p,q)} = U_{(p,q)} e^{-mv^2/2g_s},$$

has the same property, for any value of m. This corresponds geometrically to a $\mathrm{SL}(2,\mathbb{Z})$ transformation that leaves the shrinking one-cycle of the boundary T^2 invariant $T:\ b \to b$, but affects the finite cycle $a \to a + mb$. The resulting ambiguity is related to the choice of framing in Chern-Simons theory, and affects the vacuum expectation value by an overall phase that one can readily calculate, so presents no loss of predictability.

In fact, we can derive the known framing dependence by using the matrix model representation. Consider the solid torus with a Wilson line in representation R. Changing framing affects the path integral $|R_v\rangle = \sum_{\omega \in \mathcal{W}} \epsilon(\omega) \delta(v' + i g_s \omega(\alpha))$ as

$$\begin{aligned} |R_v\rangle &\to e^{\frac{-mv^2}{2g_s}} |R_v\rangle \\ &= \sum_{w \in \mathcal{W}} \epsilon(w) \int dv' e^{\frac{m(v')^2}{2g_s}} |v'\rangle \delta(v' + i g_s w(\alpha)) \\ &= e^{\frac{m g_s \alpha\cdot\alpha}{2}} |R_v\rangle\,. \end{aligned} \tag{7.11}$$

Recall that α is the highest weight vector λ of representation R shifted by the Weyl vector, i.e. $\alpha = \Lambda + \rho$. From this we see that $\alpha\cdot\alpha = C_R + \rho\cdot\rho$, where C_R is the quadratic Casimir of the representation R. Note that $\rho\cdot\rho = \frac{N(N^2-1)}{12}$. Therefore, the state $|R_v\rangle$ gets multiplied by a relative phase

$$\exp(2\pi i m h_R)$$

where $h_R = \frac{C_R}{2(k+N)}$ is the conformal weight of the primary field in representation R of the corresponding WZW model. The above result is the well-known framing dependence of Wilson lines in CS theory. The remaining phase, $\exp(m g_s \rho^2/2)$, corresponds in fact to a change in the framing of the three-manifold. Namely, $g_s\rho^2/2 = -2\pi i c_{\mathrm{U}(N)}/24$ up to a constant $2\pi N^2/24$.

JHEP02(2004)010

8. Relation to $\mathcal{N}=1$ theories

Consider IIA theory compactified on $T^*\mathbf{S}^3$ with N D6 branes wrapping $\mathbf{S}^3$. At low energies, the theory in four dimensions reduces to $\mathcal{N}=1$ super Yang-Mills, and as it was shown in [12, 36] the open string amplitudes $F_{g,h}$ lead to superpotential terms in the effective four-dimensional theory of the form

$$\int d^2\theta F_{g,h}\mathcal{W}^{2g}[NhS^{h-1}],$$

where $\mathcal{W}_{\alpha\beta}$ is an $\mathcal{N}=1$ multiplet whose bottom component is the self-dual part of the graviphoton, and $S=\mathrm{Tr}\,W_\alpha W^\alpha$ is the gluino superfield. Notice that the derivative with respect to S of the prepotential $F_0(S)=\sum_h F_{0,h}S^h$ gives the superpotential of the $\mathcal{N}=1$ theory.

The small S behavior of the superpotential is captured by the leading piece of $F_0(S)$, in other words, by the behavior of the prepotential near the conifold point

$$F_0(S)=\frac{1}{2}S^2\log S.$$

As shown in [36], this leads to the Veneziano-Yankielowicz gluino superpotential. The full prepotential is given by

$$F_0(S)=\frac{1}{2}S^2\log S+\sum_{h=4}^{\infty}\frac{B_{h-2}}{(h-2)h!}S^h,$$

and leads to a superpotential that can be written (in string units) as [36]

$$W=\sum_{n\in\mathbb{Z}}(S+2\pi in)\log(S+2\pi in)^{-N}. \tag{8.1}$$

This can be interpreted in terms of infinitely many species of domain walls labeled by n [36]. With the results of this paper we can give another interpretation of (8.1). According to the general result of [2, 37], the effective superpotential of an $\mathcal{N}=1$ supersymmetric gauge theory can be computed by a matrix model whose potential is the tree level superpotential of the gauge theory. On the other hand, we have seen that there is a hermitean matrix model describing Chern-Simons theory on the three-sphere, given by (5.22). This means that (8.1), which includes infinitely many domain walls, can be interpreted as the effective superpotential of an $\mathcal{N}=2$ theory whose tree-level superpotential is

$$\frac{1}{2}\mathrm{Tr}\,\Phi^2+S\sum_{k=0}^{\infty}\frac{B_{2k}}{(2k)(2k)!}\sum_{s=0}^{2k}(-1)^{s-1}\binom{2k}{s}\frac{1}{N}\mathrm{Tr}\,\Phi^s\,\mathrm{Tr}\,\Phi^{2k-s}. \tag{8.2}$$

Here, Φ is the $\mathcal{N}=1$ chiral superfield in the adjoint representation which is part of the $\mathcal{N}=2$ vector multiplet, and we have used that $g_s\to S/N$ [36, 2, 37]. Notice that this superpotential contains multi-trace operators. These kinds of operators have been recently considered in the context of the AdS/CFT correspondence, see for example [38, 39].

A similar argument can be applied to type-IIA theory compactified on T^*M_p, with N D6 branes wrapping M_p. Since we are orbifolding with a $\mathbb{Z}_p$ action, we have in general a quiver theory with p nodes and gauge groups $\mathrm{U}(N_1) \times \cdots \times \mathrm{U}(N_p)$. Each of these quiver theories (i.e. the different choices of $N_1, \ldots, N_p$) are in one-to-one correspondence with the choices of flat connections in the corresponding Chern-Simons theory. At leading order in the gluino superfields, this theory is just a direct product of $\mathrm{U}(N_i)$ theories that do not interact with each other, and the prepotential is just the sum of the corresponding prepotentials for the different gauge groups. However, as we have seen in this paper, the higher order corrections mix the different gluino fields, and we can interpret the resulting $\mathcal{N}=1$ superpotential as coming from a product of p $\mathcal{N}=2$ theories with gauge groups $\mathrm{U}(N_1), \ldots, \mathrm{U}(N_p)$, and with a tree level superpotential that can be read from (5.37):

$$\frac{1}{2}\sum_{i=1}^{p} \operatorname{Tr}\Phi_i^2 + \frac{S}{pN}\sum_{i=1}^{p} V(\Phi_i) - \frac{S}{pN}\sum_{1\le i<j\le p} W(\Phi_i,\Phi_j)\,,$$

where Φ_i is the $\mathcal{N}=1$ chiral superfield in the adjoint of the $\mathrm{U}(N_i)$ theory, $V(\Phi)$ is given by the second term in (8.2), and $W(\Phi_i,\Phi_j)$ is given in (5.28).

Acknowledgments

We would like to thank E. Diaconescu, R. Dijkgraaf, M. Douglas, S. Garoufalidis, J. Gomis, R. Gopakumar, A. Grassi, S. Gukov, G. Moore, B. Pioline, S. Sinha, and S. Theisen for useful discussions. The work of M.A., M.M. and C.V. is supported in part by NSF grants PHY-9802709 and DMS-0074329. A.K. is supported in part by the DFG grant KL-1070/2-1.

A. Computation of correlation functions in the gaussian matrix model

In order to compute the perturbation expansion of (5.31), one has to evaluate correlation functions in the gaussian matrix model. In this short appendix we review some basic techniques to do these computations.

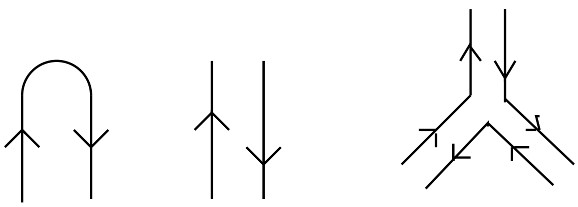


Figure 10: Fatgraphs representing $\operatorname{Tr} M$, $\operatorname{Tr} M^2$ and $\operatorname{Tr} M^3$.

We want to evaluate normalized correlation functions of the form

$$\left\langle \prod_j (\operatorname{Tr} M^j)^{k_j} \right\rangle = \frac{\int dM \mathrm{e}^{-\frac{1}{2}\operatorname{Tr} M^2} \prod_j (\operatorname{Tr} M^j)^{k_j}}{\int dM \mathrm{e}^{-\frac{1}{2}\operatorname{Tr} M^2}}\,, \tag{A.1}$$

where M is an $N \times N$ hermitean matrix. When the exponent of the gaussian is given by $-\frac{1}{2\hat{g}_s}\operatorname{Tr} M^2$, the above correlation functions gets multiplied by $\hat{g}_s^{\ell/2}$, where $\ell = \sum_j j k_j$. Notice that the correlation function is different from zero only when ℓ is even.

There are various ways to obtain the value of (A.1). A useful technique is to use the matrix version of Wick's theorem, or its graphic implementation in terms of fatgraphs (see [40] for a nice review). An insertion of $(\mathrm{Tr}\, M^j)^{k_j}$ leads to k_j j-vertices written in the double line notation, and the average (A.1) is evaluated by performing all the contractions. The propagator is the usual double line propagator. Each resulting graph Γ gives a power of N^ℓ, where ℓ is the number of closed loops in Γ. Since we have insertions of $\mathrm{Tr}\, M$ as well, we have to consider a one-vertex given by a double line in which two of the ends have been joined. The one, two and three-vertices in terms of fatgraphs are shown in figure 10.

As an example, consider the average

$$\langle \sigma_2 \rangle = \langle N\, \mathrm{Tr}\, M^4 - 4\, \mathrm{Tr}\, M\, \mathrm{Tr}\, M^3 + 3(\mathrm{Tr}\, M^2)^2 \rangle\,. \qquad \text{(A.2)}$$

The evaluation of $\langle \mathrm{Tr}\, M^4 \rangle$ is standard [24]: we have one planar diagram with weight 2 giving $2N^3$, and one nonplanar diagram (with $g = 1$) giving N. In the evaluation of $\langle \mathrm{Tr}\, M\, \mathrm{Tr}\, M^3 \rangle$ we have three possible contractions between the one-vertex and the three-vertex of figure 10, leading to a planar diagram with weight 3, as shown in figure 11. Since there are two closed loops, the final result is $3N^2$.

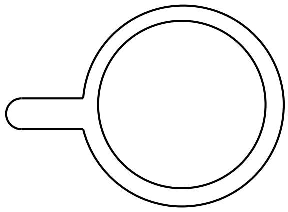
3 

Figure 11: The fatgraph contributing to $\langle \mathrm{Tr}\, M\, \mathrm{Tr}\, M^3 \rangle$.

To evaluate $\langle (\mathrm{Tr}\, M^2)^2 \rangle$ we consider two two-vertices. We can do self contractions, leading to one disconnected planar diagram with four loops, or we can contract the two-vertices one to another in two ways, leading to a connected diagram with two loops, see figure 12. We find in total $N^4 + 2N^2$. Putting everything together, we obtain:

$$\langle \sigma_2 \rangle = 5N^2(N^2 - 1)\,.$$

It turns out that one can write a general an explicit expression for the average (A.1) using results of Itzykson and Zuber [41]. This goes as follows. By Frobenius formula, one can express the product of traces $\prod_j (\mathrm{Tr}\, M^j)^{k_j}$ as a linear combination of traces in irreducible representations R. To do that, one regards the vector $(k_1, k_2, \cdots)$ as a conjugacy class of the symmetric group of $\ell = \sum_j j k_j$ elements. This conjugacy class, that we will denote by $C(\vec{k})$, has k_1 cycles of length 1, k_2 cycles of length 2, and so on. We then have,

$$\prod_j (\mathrm{Tr}\, M^j)^{k_j} = \sum_R \chi_R(C(\vec{k}))\, \mathrm{Tr}_R\, M\,, \qquad \text{(A.3)}$$

where the sum is over representations of the symmetric group of ℓ elements. These representations are associated to Young tableaux with ℓ boxes, and we will denote the number

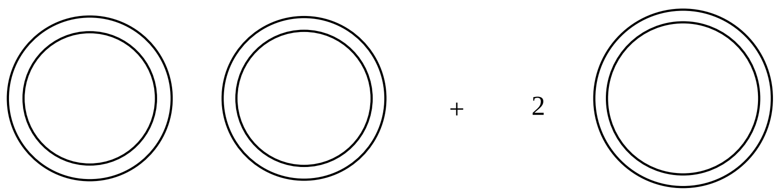


Figure 12: The fatgraphs contributing to $\langle (\mathrm{Tr}\, M^2)^2 \rangle$.

of boxes in the i-th row of the Young tableau by l_i, with $l_1 \geq l_2 \geq \cdots$. Define now the ℓ integers f_i as follows

$$f_i = l_i + \ell - i\,, \qquad i = 1, \ldots, \ell\,. \tag{A.4}$$

We will say that the Young tableau associated to R is even if the number of odd f_i's is the same as the number of even f_i's. Otherwise, we will say that it is odd (remember that ℓ is even). One can show [41] that the average of $\mathrm{Tr}_R\, M$ in the gaussian matrix model vanishes if R is an odd tableau, and for even tableaux one has the explicit formula:

$$\langle \mathrm{Tr}_R\, M\rangle = (-1)^{\frac{A(A-1)}{2}} \frac{\prod_{f\,\mathrm{odd}} f!! \prod_{f'\,\mathrm{even}} f'!!}{\prod_{f\,\mathrm{odd},f'\,\mathrm{even}}(f-f')} d_R\,, \tag{A.5}$$

where $A = \ell/2$. Here d_R is the dimension of R as an irreducible representation of $\mathrm{U}(N)$, and can be computed for example by using the hook formula. As an example of (A.5), let us compute $\langle \mathrm{Tr}\, M\, \mathrm{Tr}\, M^3\rangle$. To do that, one has to evaluate $\langle \mathrm{Tr}_R\, M\rangle$ for $R = \yng(4)$, $\yng(2,2)$ and $\yng(1,1,1,1)$. All of these tableaux are even, and one finds:

$$\begin{aligned}
\langle \mathrm{Tr}_{\yng(4)}\, M\rangle &= \frac{1}{8} N(N+1)(N+2)(N+3)\,,\\
\langle \mathrm{Tr}_{\yng(2,2)}\, M\rangle &= \frac{1}{4} N^2(N^2-1)\,,\\
\langle \mathrm{Tr}_{\yng(1,1,1,1)}\, M\rangle &= \frac{1}{8} N(N-1)(N-2)(N-3)\,.
\end{aligned} \tag{A.6}$$

One then finds, by using Frobenius formula,

$$\langle \mathrm{Tr}\, M\, \mathrm{Tr}\, M^3\rangle = \langle \mathrm{Tr}_{\yng(4)}\, M\rangle - \langle \mathrm{Tr}_{\yng(2,2)}\, M\rangle + \langle \mathrm{Tr}_{\yng(1,1,1,1)}\, M\rangle = 3N^2\,, \tag{A.7}$$

in agreement with the result that we obtained with fatgraphs.

Although the result of [41] explained above gives a general answer, in some cases there are more convenient expressions. For example, for $\langle \mathrm{Tr}\, M^{2j+2}\rangle$, Kostov and Mehta [42] found the useful result:

$$\langle \mathrm{Tr}\, M^{2j+2}\rangle = \frac{(2j+2)!}{(j+1)!(j+2)!} P_{j+1}(N)\,, \tag{A.8}$$

where

$$P_m(N) = \sum_{i=0}^{[m/2]} \frac{a_{mi}}{4^i} N^{m+1-2i}\,, \tag{A.9}$$

and the coefficients a_{mi} are defined by the recursion relation

$$a_{m+1,i} = \sum_{k=2i-1}^{m} k(k+1) a_{k-1,i-1}\,, \tag{A.10}$$

and $a_{m0} = 1$. One has for example $a_{m1} = (m+1)m(m-1)/3$, and so on. Notice that in the planar limit, the leading term of the average (A.8) is given by N^{j+2} times the Catalan number $c_{j+1} = (2j+2)!/((j+1)!(j+2)!)$. Using (A.8), one finds for example,

$$\begin{aligned}\langle \mathrm{Tr}\, M^4 \rangle &= 2N^3 + N\,, \\ \langle \mathrm{Tr}\, M^6 \rangle &= 5N^4 + 10N^2\,, \\ \langle \mathrm{Tr}\, M^8 \rangle &= 14N^5 + 70N^3 + 21N\,.\end{aligned}$$

Finally, another useful fact in the computation of (A.1) is that averages of the form $\langle (\mathrm{Tr}\, M^2)^p\, \mathcal{O}\rangle$ can be evaluated by restoring appropriately the $g = 1/\hat{g}_s$ dependence in the gaussian. One easily finds that, if $\mathcal{O}$ is an operator of the form $\prod_j (\mathrm{Tr}\, M^j)^{k_j}$, with $\ell = \sum_k jk_j$, then

$$\langle (\mathrm{Tr}\, M^2)^p\, \mathcal{O}\rangle = \left(-2\frac{d}{dg}\right)^p g^{-\frac{\ell+N^2}{2}}\Big|_{g=1} \langle \mathcal{O}\rangle\,.$$

B. Derivation of the propagators in the B-model

One main problem in the analysis of the B-model is the determination of the propagators S^{ij} with the defining relation $\bar{\partial}_{\bar{i}} S^{kl} = \bar{C}_{\bar{i}}^{kl}$ [12]. They are simply integrated w.r.t. $\bar{\partial}_{\bar{\jmath}}$ from the special geometry relation

$$R_{i\bar{\jmath}l}^k = G_{i\bar{\jmath}}\delta_l^k + G_{k\bar{\jmath}}\delta_i^k - C_{ilm}\bar{C}_{\bar{\jmath}}^{km}\,, \qquad \text{(B.1)}$$

using the wellknown formulas in Kähler geometry $R_{i\bar{\jmath}l}^k = -\bar{\partial}_{\bar{\jmath}}\Gamma_{il}^k$, $G_{i\bar{\jmath}} = \partial_i\bar{\partial}_{\bar{\jmath}}K$ and $\Gamma_{lm}^i = G^{i\bar{k}}\partial l G_{\bar{k},m}$ to

$$S^{ij}C_{jkl} = \delta_l^i\partial_k K + \delta_k^i\partial_l K + \Gamma_{kl}^i + f_{kl}^i\,. \qquad \text{(B.2)}$$

However there are two problems in actually solving for the S^{ij}. The purely holomorphic terms f_{kl}^i are ambiguous integration constants. In the multi moduli case the $\frac{1}{2}n^2(n+1)$ equations overdetermine the $\frac{1}{2}n(n+1)$ $S^{ji} = S^{ji}$, $i \leq j$ and the f_{kl}^i can in general not be trivial. Secondly, since the left hand side of (B.2) is covariant, the f_{kl}^i have to undo the inhomogeneous transformation of Christoffel symbol as well as the shift of the first two terms of the left hand side under Kähler transformations.

While for the instanton expansion we need the flat large complex structure variables $t_i(z_k)$, we expect the f_{kl}^i to be simple rational functions involving the discriminant components in the z_i variables, because the C_{ikl} have similar properties in these coordinates. We will first solve the problem of the over determination of the S^{ij} in the z coordinates and then transform the S^{ij} as covariant tensors to the t variables, this determines the choice of the f_{jk}^i in the t variables.

Let us discuss some non-compact Calabi-Yau manifolds first. Here we have the simplification that in the holomorphic limit the Kähler potential becomes a constant and furthermore there is a gauge [33] in which the propagators $\bar{\partial}_{\bar{\jmath}} S^j := S_{\bar{\jmath}}^j$ and $\bar{\partial}_{\bar{\jmath}} S := S_{\bar{\jmath}}$ vanish in that limit, which makes the topological amplitudes entirely independent from quantities like the Euler number or the Chern classes, which would have to be regularized in the non-compact case.

The simplest cases to consider are $\mathcal{O}(-2,-(n+2)) \to F_n$. We use the parameterization of the complex structure variables of [28, 29], where the Picard-Fuchs equations and the genus zero and genus one results can be found. For $n=0$, i.e. $F_0 = \mathbb{P}^1 \times \mathbb{P}^1$ we note that the threepoint functions are given in the z variables by

$$C_{111} = \frac{\Delta_2 - 16 z_1(1+z_1)}{4 z_1^3 \Delta}, \qquad C_{112} = \frac{16 z_1^2 - \Delta_2}{4 z_1^2 z_2 \Delta}, \qquad C_{122} = \frac{16 z_2^2 - \Delta_1}{4 z_1 z_2^2 \Delta}, \tag{B.3}$$

where $\Delta_i = (1-4z_i)$ and

$$\Delta = 1 - 8(z_1+z_2) + 16(z_1-z_2)^2$$

is the conifold discriminant. Other three-point function follow for F_0 by symmetry. Generally these couplings of the local models can be obtained from the compact elliptic fibration over F_n with fiber $X_6(1,2,3)$ by a limiting procedure. This compact Calabi-Yau has three complexified volumes: t_E, roughly the volume of the elliptic fiber, and t_B and t_F the volume of base and the fiber of the Hirzebruch surface F_n. They correspond to the Mori cone generators $(-6,3,2,1,0,0,0,0)$, $(0,0,0,(n-2),1,1,-n,0)$, $(0,0,0,-2,0,0,1,1)$ and fulfill in large complex structure limit $\log(z_a) = t_a$. It turns out that the limit of is not given by $t_E \to \infty$ but rather by $\widetilde{t}_E = (t_E - \frac{K\cdot B}{8} t_B - \frac{K\cdot F}{8} t_F) \to \infty$

With these couplings we find a particular solution to (B.2) by choosing

$$\begin{aligned} f^1_{12} &= -\frac{1}{4z_2}, \qquad & f^2_{12} &= -\frac{1}{4z_1}, \\ f^1_{11} &= -\frac{1}{z_1}, \qquad & f^2_{22} &= -\frac{1}{z_2}, \end{aligned} \tag{B.4}$$

where the rest are either related by symmetry to the above or zero. Note that this simple choice of the integration constants implies algebraic relations between the Christoffel symbols in the z coordinates in the holomorphic limit

$$\lim_{\bar z \to 0} \Gamma^{z_a}_{z_b z_c} = \lim_{\bar z \to 0} G^{z_a \bar z_e} \partial_{z_b} G_{\bar z_e, z_c} = \frac{\partial z_a}{\partial t_e} \frac{\partial}{\partial z_b} \frac{\partial t_e}{\partial z_c}.$$

These relations are due the fact that only one transcendental mirror map exists. In particular the following relation between the mirror maps is easily shown from the Picard-Fuchs equations

$$\frac{z_1}{z_2} = \frac{q_1}{q_2} \tag{B.5}$$

with $q_i = e^{-t_i}$. With this we can obtain the general solution to the integrability constraints as a rational relation between the f^i_{jk} as

$$\begin{aligned} f^1_{11} &= \frac{6z_s - 1}{z_1(1-4z_s)} + \frac{8 f^1_{12} z_2 z_s}{z_1(1-4z_s)} - f^1_{22} \frac{z_2^2}{z_1^2}, \\ f^2_{11} &= \frac{z_2(6z_s - 1)}{z_1^2(1-4z_s)} + \frac{8 f^2_{12} z_2 z_s}{z_1(1-4z_s)} - f^2_{22} \frac{z_2^2}{z_1^2}, \end{aligned}$$

where $z_s = z_1 + z_2$.

Claim: The remaining degrees of freedom in the choice of f^i_{jk} can always be used to set all but one S^{kk} to zero. This has been checked for F_0, see also [43], F_1 and F_2 and is probably true more generally.[10]

In view of (B.5) we can see this most easily for F_0 in the variables

$$\begin{aligned} z &= \frac{z_1}{z_2}\,, \qquad q = \frac{q_1}{q_2}\,, \\ Z &= z_2\,, \qquad Q = q_2\,, \end{aligned} \tag{B.6}$$

in which some Christoffel symbols are rational

$$\Gamma^1_{11} = \frac{1}{z}\,, \qquad \Gamma^1_{12} = 0\,, \qquad \Gamma^1_{22} = 0$$

We can set $f^1_{11} = -\frac{1}{z}$, $f^1_{12} = 0$ and $f^1_{22} = 0$ so that by

$$S^{ik} = (C_p^{-1})^{kl}(\Gamma^i_{pl} + f^i_{pl})$$

we have $S^{1p} = 0$. Because of $S^{2k} = (C_p^{-1})^{kl}(\Gamma^2_{pl} + f^2_{pl})$ we must ensure that there is a rational relation between the Γ^2_{kl} and rational choice of f^2_{kl}, which is compatible with $S^{12} = S^{21} = 0$. A particular choice corresponding to (B.4) is given by $f^1_{11} = 0$, $f^2_{12} = -\frac{1}{4z}$ and $f^2_{22} = -\frac{3}{2Z}$.

We need the propagator in the local orbifold coordinates. It follows from the tensorial transformation law of the left hand side of (B.2) and the transformation of the Christoffel on the right that a possible choice of the ambiguity at the orbifold is given by $\widetilde{f}^a_{bc} = \frac{\partial x_a}{\partial z_l}\left(\frac{\partial z_l}{\partial x_b \partial x_c}\right) + \frac{\partial x_a}{\partial z_j}\frac{\partial z_k}{\partial x_b}\frac{\partial z_l}{\partial x_c} f^j_{kl}$, where f^i_{kl} are the ambiguities (B.4) and the transformation is given by (6.3). This formula holds since the s_i and t_i are related by a $GL(2,\mathbb{C})$ transformation and yields $\widetilde{f}^1_{11} = \frac{1}{1-x_1}$, $\widetilde{f}^1_{12} = \widetilde{f}^1_{22} = \widetilde{f}^2_{22} = 0$, $\widetilde{f}^2_{11} = -\frac{x_2}{2x_1(1-x_1)}$ and $\widetilde{f}^2_{12} = \frac{4-3x_1}{4x_1(1-x_1)}$. Note that $\Gamma^1_{11} = -\frac{1}{(1-x_1)}$ in the x coordinates and we get $S^{11} = S^{12} = S^{21} = 0$. The only nonvanishing propagator in the s_1, s_2 coordinates is

$$S^{22} = \frac{1}{16}(s_2 - s_1)(s_1 + s_2) + \frac{1}{6144}\left((s_1 - s_2)(s_1 + s_2)(s_1^2 - 5s_2^2)\right) + O(s^6)\,. \tag{B.7}$$

The fact that only one propagator contributes allows a consistency check or an alternative way of deriving the propagator. Namely by noting that the holomorphic anomaly equation for the genus 1 partition function can be either derived using the contact terms in topological field theory or more geometrically via a generalized determinant calculation similar as in Quillen's work

$$\partial_i \bar\partial_{\bar j} F^{(1)} = \frac{1}{2} C_{ikl} \bar C_{\bar j \bar k \bar l} e^{2K} G^{\bar k k} G^{l \bar l} - \left(\frac{\chi}{24} - 1\right) G_{i\bar j}\,. \tag{B.8}$$

If we specialize this to the non-compact case the last term becomes irrelevant in the holomorphic limit. Using the definition of the propagator $\bar\partial_{\bar j} S^{kl} := \bar C_{\bar j \bar k \bar l} e^{2K} G^{\bar k k} G^{l \bar l}$ and the fact that C_{ikl} is truly holomorphic we may write this as

$$\bar\partial_{\bar j}\left[\partial_i F^1 - \frac{1}{2} C_{ikl} S^{kl}\right] = 0\,. \tag{B.9}$$

This is the easiest example of the Feynman graph expansion of the anomaly equation, see figure 13.

[10]It would be interesting to check for compact Calabi-Yau.

The result is that $F^{(1)}$ can be integrated in the holomorphic limit from

$$\partial_{t_i} F^{(1)} = S^{jk} \partial_i \partial_j \partial_k F^{(0)} + \partial_{t_i} \sum_{r=1}^{s} a_r \log(\Delta_r)\,, \qquad \text{(B.10)}$$

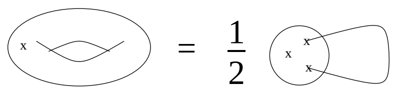


Figure 13: Degeneration of the marked torus.

where $\Delta_r = 0$ are the various singular divisors in the moduli space and $\sum_{r=1}^{s} a_r \log(\Delta_r)$ parameterize the holomorphic ambiguity. Since only S^{22} is nonzero we can invert the equation (B.10) and obtain (B.7) from the knowledge of $F^{(0)}, F^{(1)}$. A singular behavior of S^{22} at the discriminant can be absorbed by choosing the a_r appropriately. In our case the only nonzero a_r will be $a_{con} = \frac{1}{12}$ in order to recover the previous gauge choice (B.7).[11] We have fixed the holomorphic ambiguity up to genus three. Using the transformation properties of the ambiguities this allows to calculate $F^{(g)}$, $g = 0, 1, 2, 3$ at all points in the moduli space. We checked that the expansion at large complex structure matches the Gromov-Witten invariants in [8] and [43].

References

[1] M. Mariño, *Chern-Simons theory, matrix integrals and perturbative three-manifold invariants*, hep-th/0207096.

[2] R. Dijkgraaf and C. Vafa, *Matrix models, topological strings and supersymmetric gauge theories*, *Nucl. Phys.* **B 644** (2002) 3 [hep-th/0206255].

[3] E. Witten, *Chern-Simons gauge theory as a string theory*, *Prog. Math.* **133** (1995) 637–678 [hep-th/9207094]; in *The Floer memorial volume*, H. Hofer, C.H. Taubes, A. Weinstein and E. Zehner, eds., Birkhäuser 1995, p. 637.

[4] R. Gopakumar and C. Vafa, *On the gauge theory/geometry correspondence*, *Adv. Theor. Math. Phys.* **3** (1999) 1415 [hep-th/9811131].

[5] H. Ooguri and C. Vafa, *Worldsheet derivation of a large-N duality*, *Nucl. Phys.* **B 641** (2002) 3 [hep-th/0205297].

[6] A. Giveon, A. Kehagias and H. Partouche, *Geometric transitions, brane dynamics and gauge theories*, *J. High Energy Phys.* **12** (2001) 021 [hep-th/0110115].

[7] M. Kontsevich, *Intersection theory on the moduli space of curves and the matrix airy function*, *Commun. Math. Phys.* **147** (1992) 1.

[8] M. Aganagic, M. Mariño and C. Vafa, *All loop topological string amplitudes from Chern-Simons theory*, hep-th/0206164.

[9] D.-E. Diaconescu, B. Florea and A. Grassi, *Geometric transitions and open string instantons*, *Adv. Theor. Math. Phys.* **6** (2003) 619 [hep-th/0205234]; *Geometric transitions, del Pezzo surfaces and open string instantons*, *Adv. Theor. Math. Phys.* **6** (2003) 643 [hep-th/0206163].

[10] M. Aganagic and C. Vafa, *G_2 manifolds, mirror symmetry and geometric engineering*, hep-th/0110171.

[11] E. Brezin, C. Itzykson, G. Parisi and J.B. Zuber, *Planar diagrams*, *Commun. Math. Phys.* **59** (1978) 35.

[11] In fact one can drop the ambiguity part $\partial_{t_i} \sum_{r=1}^{s} a_r \log(\Delta_r)$ in (B.10) altogether. This corresponds merely to gauge choice of the propagator which leads to a different form of the ambiguity at higher genus.

[12] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, *Kodaira-spencer theory of gravity and exact results for quantum string amplitudes*, *Commun. Math. Phys.* **165** (1994) 311 [hep-th/9309140].

[13] E. Witten, *Quantum field theory and the Jones polynomial*, *Commun. Math. Phys.* **121** (1989) 351.

[14] S. Elitzur, G.W. Moore, A. Schwimmer and N. Seiberg, *Remarks on the canonical quantization of the Chern-Simons-Witten theory*, *Nucl. Phys.* **B 326** (1989) 108.

[15] J.M.F. Labastida and A.V. Ramallo, *Operator formalism for Chern-Simons theories*, *Phys. Lett.* **B 227** (1989) 92;
J.M.F. Labastida, P.M. Llatas and A.V. Ramallo, *Knot operators in Chern-Simons gauge theory*, *Nucl. Phys.* **B 348** (1991) 651.

[16] L. Rozansky, *A large k asymptotics of Witten's invariant of Seifert manifolds*, *Commun. Math. Phys.* **171** (1995) 279 [hep-th/9303099]; *A contribution to the trivial connection to Jones polynomial and Witten's invariant of 3-d manifolds. 1*, *Commun. Math. Phys.* **175** (1996) 275 [hep-th/9401061].

[17] S.K. Hansen and T. Takata, *Reshetikhin-Turaev invariants of Seifert 3-manifolds for classical simple Lie algebras, and their asymptotic expansions*, math.GT/0209403.

[18] K. Hori and C. Vafa, *Mirror symmetry*, hep-th/0002222.

[19] M. Aganagic and C. Vafa, *Perturbative derivation of mirror symmetry*, hep-th/0209138.

[20] M. Aganagic, A. Klemm and C. Vafa, *Disk instantons, mirror symmetry and the duality web*, Z. Naturforsch. A **57** (2002) [hep-th/0105045].

[21] E. Witten, *Branes and the dynamics of QCD*, *Nucl. Phys.* **B 507** (1997) 658, [hep-th/9706109].

[22] M. Aganagic and C. Vafa, "Mirror symmetry, D-branes and counting holomorphic discs," hep-th/0012041.

[23] R. Dijkgraaf and C. Vafa, *On geometry and matrix models*, *Nucl. Phys.* **B 644** (2002) 21, [hep-th/0207106].

[24] D. Bessis, C. Itzykson and J. B. Zuber, *Quantum field theory techniques in graphical enumeration*, *Adv. Appl. Math.* **1** (1980) 109.

[25] R. Dijkgraaf, S. Gukov, V. A. Kazakov and C. Vafa, *Perturbative analysis of gauged matrix models*, *Phys. Rev.* **D 68** (2003) 045007, [hep-th/0210238].

[26] G. Bonnet, F. David and B. Eynard, *Breakdown of universality in multi-cut matrix models*, *J. Phys.* **A 33** (2000) 6739, [cond-mat/0003324].

[27] R. Gopakumar and C. Vafa, *M-theory and topological strings, I*, hep-th/9809187.

[28] S. Katz, A. Klemm and C. Vafa, *Geometric engineering of quantum field theories*, *Nucl. Phys.* **B 497** (1997) 173, [hep-th/9609239].

[29] T.M. Chiang, A. Klemm, S.T. Yau and E. Zaslow, *Local mirror symmetry: calculations and interpretations*, *Adv. Theor. Math. Phys.* **3** (1999) 495, [hep-th/9903053].

[30] V.I. Arnold, S.M. Gusein-Zade and A.N. Varchenko, *Singularities of differentiable maps: vol II*, Birhäuser, Boston (1988).

JHEP02(2004)010

[31] P. Candelas, X. De La Ossa, A. Font, S. Katz and D.R. Morrison, *Mirror symmetry for two parameter models. I*, *Nucl. Phys.* **B 416** (1994) 481, [hep-th/9308083].

[32] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, *Holomorphic anomalies in topological field theories*, *Nucl. Phys.* **B 405** (1993) 279 [hep-th/9302103].

[33] A. Klemm and E. Zaslow, *Local mirror symmetry at higher genus*, hep-th/9906046.

[34] L.-L. Chau and O. Zaboronsky, *On the structure of correlation functions in the normal matrix model*, *Commun. Math. Phys.* **196** (1998) 203 [hep-th/9711091].

[35] H. Ooguri and C. Vafa, *Knot invariants and topological strings*, *Nucl. Phys.* **B 577** (2000) 419 [hep-th/9912123].

[36] C. Vafa, *Superstrings and topological strings at large-N*, *J. Math. Phys.* **42** (2001) 2798 [hep-th/0008142].

[37] R. Dijkgraaf and C. Vafa, *A perturbative window into non-perturbative physics*, hep-th/0208048.

[38] O. Aharony, M. Berkooz and E. Silverstein, *Multiple-trace operators and non-local string theories*, *J. High Energy Phys.* **08** (2001) 006 [hep-th/0105309].

[39] E. Witten, *Multi-trace operators, boundary conditions and AdS/CFT correspondence*, hep-th/0112258.

[40] P. Di Francesco, *Matrix model combinatorics: applications to folding and coloring*, math-ph/9911002.

[41] C. Itzykson and J.B. Zuber, *Matrix integration and combinatorics of modular groups*, *Commun. Math. Phys.* **134** (1990) 197.

[42] I.K. Kostov and M.L. Mehta, *Random surfaces of arbitrary genus: exact results for $D = 0$ and -2 dimensions*, *Phys. Lett.* **B 189** (1987) 118.

[43] S. Hosono, *Counting BPS states via holomorphic anomaly equations*, hep-th/0206206.